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Classical *R*-matrix theory for bi-Hamiltonian field systems

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Abstract
This is a survey of the application of the classical *R*-matrix formalism to the construction of infinite-dimensional integrable Hamiltonian field systems. The main point is the study of bi-Hamiltonian structures. Appropriate constructions on Poisson, noncommutative and loop algebras as well as the central extension procedure are presented. The theory is developed for (1+1)- and (2+1)-dimensional field and lattice soliton systems as well as hydrodynamic systems. The formalism presented contains sufficiently many proofs and important details to make it self-contained and complete. The general theory is applied to several infinite-dimensional Lie algebras in order to construct both dispersionless and dispersive (soliton) integrable field systems.

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1. Introduction

Finding a systematic method for the construction of integrable nonlinear systems is one of the most important issues in the theory of evolutionary systems. A very powerful tool called the classical *R*-matrix formalism [81] has proved to be very fruitful in the systematic construction of the field and lattice soliton systems as well as dispersionless systems. The crucial point of the formalism is the observation that integrable dynamical systems can be obtained from the Lax equations on appropriate Lie algebras. Besides, a huge part of integrable field systems possessing Lax representation can be obtained within the classical *R*-matrix formalism, if not all of them. The greatest advantage of this formalism, besides the possibility of systematic construction of the integrable systems, is the construction of bi-Hamiltonian structures and (infinite) hierarchies of symmetries and conserved quantities.

The ideas of the *R*-matrix theory date back to the world renowned 'St Petersburg School' represented by Faddeev, Reyman, Sklyanin, Semenov-Tian-Shansky and others (see [31] and references therein). The abstract formalism of classical R-matrices appeared in the paper [86] by Sklyanin as an intermediate step within the inverse quantum scattering method and was further developed by Belavin and Drinfel'd [6, 26]. The present version of the formalism, together with definition 2.1 of the classical R-matrix, was given by Semenov-Tian-Shansky in [81].

One of the most characteristic features of integrable nonlinear systems is the existence of bi-Hamiltonian structures. This ingenious concept was introduced by Magri [58] in 1978. From the geometrical point of view, this means that there exists a pair of compatible Poisson tensors which allow us, using a recursion chain, to generate infinite (in the infinite-dimensional case) hierarchies of commuting symmetries and constants of motion being in involution with respect to the above Poisson tensors. In order to stress the importance of the bi-Hamiltonian structures for evolution systems let us quote Dickey[3].

> The existence of two compatible Poisson (or Hamiltonian) structures is a remarkable feature of most, if not all, integrable systems, sometimes it is considered as the essence of the integrability.

For the theory of infinite-dimensional bi-Hamiltonian systems we refer the reader to the following references [8, 23, 25, 68].

The goal of this paper is to present an introductory survey of the formalism of classical R-matrices applied to infinite-dimensional Lie algebras in order to construct integrable systems with infinitely many degrees of freedom and related Hamiltonian and bi-Hamiltonian structures.

In the first part of the paper (section 2), we present in a systematic fashion the concept of classical R-matrix formalism with many proofs and important details to make the text self-contained and complete. First of all we present the basics of the formalism, where concepts of Lax hierarchies, Lie–Poisson structures and ad-invariant scalar products are explained. Next, we present the construction of integrable hierarchies with multi-Hamiltonian structures on Poisson and noncommutative algebras, respectively. Then, we show how to extend the whole formalism via the so-called central extension procedure. Finally, we apply the classical R-matrix formalism to loop algebras taking also into consideration the central extension approach.

We would like to point out that the structure of section 2 reflects the applications of the classical R-matrix formalism in the following sections. Besides, its content is chosen in such a way that the reader not familiar with the theory presented could fully understand all results of sections 3 and 4 without any additional assistance.

In the second part of the paper, we apply the formalism developed in the first part to several important Lie algebras with the aim of the construction of (1+1)- and (2+1)-dimensional integrable hierarchies together with their Hamiltonian (multi-Hamiltonian) representation. Section 3 deals with dispersionless or equivalently hydrodynamic systems, where two kinds of algebras are considered. As the first case, we consider the case of Poisson algebras and related bi-Hamiltonian dispersionless systems. As the second case, we consider the so-called universal hierarchy, based on the Lie algebra of vector fields on the circle, and related dispersionless systems. In both cases the central extension approach is also applied. In section 4, the construction of soliton hierarchies, i.e. integrable systems with dispersion, is presented. What is important, in the first two subsections the theory is developed in such a way that it covers in a single unified formalism not only standard lattice and field soliton systems, but also q-deformed ones. The Lie algebras of shift operators and generalized pseudo-differential

[3] [23], p 43.

operators are used here. Finally, the application of the classical R-matrix formalism to the loop algebras is illustrated on the example of $sl(2, \mathbb{C})$ semi-simple Lie algebra. All the above applications and related Lie algebras lead to the construction of a huge number of specific integrable systems among which we present explicitly only a few of them, referring the reader to the cited references for further examples.

In the following survey, classical R-matrix theory is presented in the framework of an appropriate infinite-dimensional Lie algebra, with the aim of construction of integrable bi-Hamiltonian systems. However, the formalism is significantly more powerful as for example it is intimately connected with factorization and Riemann–Hilbert problems for the related Lie groups [81]. Besides we completely neglect here the theory of finite-dimensional systems, thus for more information and the complete theory of classical R-matrices we send the reader to the original papers [78, 79, 81] as well as reviews [63, 82, 83] and book [31].

2. Classical *R*-matrix theory

In this section, we will present a unified approach to the construction of integrable evolution equations together with their (multi-)Hamiltonian structures. The idea originates from the pioneering article [40] by Gelfand and Dickey, where they presented a construction of Hamiltonian soliton systems of KdV type using pseudo-differential operators. Next, Adler [2] showed how to construct the bi-Hamiltonian structures for the above soliton systems using the method based on the Kostant–Symes theorem obtained independently in [50, 92]. Later the abstract formalism of classical R-matrices was formulated by Semenov-Tian-Shansky [81] and further developed together with Reyman [79]. In [56, 66] it was shown that there are in fact three natural Poisson brackets associated with classical R-structures. Quite recently Li [55] considered the classical R-matrix theory on the so-called (commutative) Poisson algebras. This approach leads to the construction of multi-Hamiltonian systems of hydrodynamic (dispersionless) type.

2.1. Classical R-matrices

Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{K}$ of complex or real numbers, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, that is, $\mathfrak{g}$ is equipped with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a Lie bracket, which is skew-symmetric and satisfies the Jacobi identity. The Lie bracket $[\cdot,\cdot]$ defines the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$: $\mathrm{ad}_a b \equiv [a, b]$.

Definition 2.1 [81]. *A linear map $R : \mathfrak{g} \to \mathfrak{g}$ such that the operation*

$$[a, b]_R := [Ra, b] + [a, Rb] \qquad a, b \in \mathfrak{g} \tag{1}$$

defines another Lie bracket on $\mathfrak{g}$ is called the classical R-matrix.

The skew-symmetry of (1) is obvious. As for the Jacobi identity for (1), we find that

$$\begin{aligned} 0 &= [a, [b, c]_R]_R + \text{c.p.} = [Ra, [Rb, c]] + [Ra, [b, Rc]] + [a, R[b, c]_R] + \text{c.p.} \\ &= [Rb, [Rc, a]] + [Rc, [a, Rb]] + [a, R[b, c]_R] + \text{c.p.} \\ &= [a, R[b, c]_R - [Rb, Rc]] + \text{c.p.}, \end{aligned} \tag{2}$$

where c.p. stands for cyclic permutations within the triple $\{a, b, c\} \in \mathfrak{g}$, and the last equality follows from the Jacobi identity for $[\cdot,\cdot]$. Hence, a sufficient condition for R to be a classical R-matrix is to satisfy the so-called (modified) Yang–Baxter equation, YB(α):

$$[Ra, Rb] - R[a, b]_R + \alpha[a, b] = 0, \tag{3}$$

where α is a number from $\mathbb{K}$. There are only two relevant cases of YB(α), namely $\alpha \neq 0$ and $\alpha = 0$, as all Yang–Baxter equations with $\alpha \neq 0$ are equivalent up to a rescaling of α.

Definition 2.2. *A linear operator* $A : \mathfrak{g} \to \mathfrak{g}$ *is called intertwining if* $A \circ \mathrm{ad}_a = \mathrm{ad}_a \circ A$*, i.e., if* $A[a, b] = [Aa, b] = [a, Ab]$ *for any* $a, b \in \mathfrak{g}$*.*

Proposition 2.3 [79]**.** *If* R *is a classical R-matrix and* A *is an intertwining operator that is nondegenerate, i.e.* $\ker A = 0$*, then* $R \circ A$ *also is a classical R-matrix.*

Proof. We have

$$\begin{aligned} RA[a, b]_{RA} &= RA[RAa, b] + RA[a, RAb] \\ &= R[RAa, Ab] + R[Aa, RAb] = R[Aa, Ab]_R. \end{aligned}$$

Hence,

$$[RAa, RAb] - RA[a, b]_{RA} = [RAa, RAb] - R[Aa, Ab]_R$$

and the Jacobi identity for $[\cdot, \cdot]_{RA}$ with respect to the elements $a, b, c \in \mathfrak{g}$ reduces to the Jacobi identity for $[\cdot, \cdot]_R$ with respect to the elements Aa, Ab, Ac, see (2). □

In fact, the nondegeneracy condition of A in the above proposition can be omitted, see [79]. Note that a linear combination of intertwining operators again is an intertwining operator.

2.2. Lax hierarchy

In this section, we present the classical R-matrix formalism for the class of Lie algebras for which the Lie bracket additionally satisfies the Leibniz rule. Later, while considering the loop algebras in section 2.10, we shall drop this extra condition.

Assume that the Lie algebra $\mathfrak{g}$ is also an algebra with respect to an associative multiplication such that

$$\mathrm{ad}_a(bc) = \mathrm{ad}_a(b)c + b\,\mathrm{ad}_a(c) \qquad \Longleftrightarrow \qquad [a, bc] = [a, b]c + b[a, c] \tag{4}$$

the Leibniz rule holds, i.e., the Lie bracket $[\cdot, \cdot]$ is a derivation with respect to the multiplication. Note that this condition is satisfied automatically in the case of a commutative algebra $\mathfrak{g}$ when the Lie bracket is given by a finite-dimensional Poisson bracket, as well as in the case of a non-commutative algebra $\mathfrak{g}$ with the Lie bracket given by the commutator.

In the construction of integrable hierarchies an important role is played by smooth maps $X : \mathfrak{g} \to \mathfrak{g}$, $L \mapsto X(L)$ being invariants of the Lie bracket, or equivalently ad-invariant, that is they are such that

$$\mathrm{ad}_L X(L) = 0 \qquad \Longleftrightarrow \qquad [X(L), L] = 0. \tag{5}$$

The smoothness of the mapping X means that its differential and directional derivatives exist and are well defined.

As a consequence of the above assumption (4), any map $X(L)$ being differentiable map of a single variable L is an invariant: $[X(L), L] = 0$, since it is assumed that the adjoint action of the Lie bracket is a derivation of the associative multiplication in the algebra. The natural choice for invariant smooth functions is the power functions $X_n(L) = L^n$ that are always well defined on $\mathfrak{g}$ equipped with an associative multiplication. One can consider less trivial functions, for example the logarithmic ones, like $X(L) = \ln L$, but only when they have proper interpretation in $\mathfrak{g}$.

Proposition 2.4. *Smooth invariant functions $X_n(L)$ generate a hierarchy of vector fields on $\mathfrak{g}$ of the form*

$$L_{t_n} = [RX_n(L), L] \qquad L \in \mathfrak{g} \quad n \in \mathbb{Z}, \tag{6}$$

where t_n are evolution parameters. Assuming that a classical R-matrix R commutes with the directional derivatives with respect to all (6), the Yang–Baxter equation (3) is a sufficient condition for the pairwise commutativity of the vector fields (6).

Proof. The directional derivative of a smooth function $F : \mathfrak{g} \to \mathfrak{g}$ in the direction of (6) is given by

$$F(L)_{t_n} = F(L)'\big[L_{t_n}\big] = [RX_n(L), F(L)], \tag{7}$$

which follows from the Leibniz rule (4). Thus, one finds that

$$\begin{aligned}\big(L_{t_m}\big)_{t_n} - \big(L_{t_n}\big)_{t_m} &= [RX_m(L), L]_{t_n} - [RX_n(L), L]_{t_m} \\ &= \big[(RX_m(L))_{t_n} - (RX_n(L))_{t_m}, L\big] + [RX_m(L), [RX_n(L), L]] \\ &\quad - [RX_n(L), [RX_m(L), L]] \\ &= \big[(RX_m(L))_{t_n} - (RX_n(L))_{t_m} + [RX_m(L), RX_n(L)], L\big].\end{aligned} \tag{8}$$

Hence, for the pairwise commutativity of vector fields (6) it suffices that the so-called zero-curvature equations

$$(RX_m(L))_{t_n} - (RX_n(L))_{t_m} + [RX_m(L), RX_n(L)] = 0 \tag{9}$$

hold.

The assumption that R commutes with directional derivatives implies that it commutes with the derivatives with respect to evolution parameters, i.e.,

$$(RL)_{t_n} = RL_{t_n}. \tag{10}$$

Thus, the right-hand side of (9) becomes

$$\begin{aligned}&R(X_m(L))_{t_n} - R(X_n(L))_{t_m} + [RX_m(L), RX_n(L)] \\ &\quad \overset{\text{by}(7)}{=} R[RX_n(L), X_m(L)] - R[RX_m(L), X_n(L)] + [RX_m(L), RX_n(L)] \\ &\quad = [RX_m(L), RX_n(L)] - R[X_m(L), X_n(L)]_R.\end{aligned} \tag{11}$$

Now, if the R-matrix R satisfies the Yang–Baxter equation (3) then the last expression is equal to $-\alpha[X_m(L), X_n(L)] = 0$, and the result follows. □

The hierarchy (6) is called *the Lax hierarchy* and L is called the Lax operator or the Lax function depending on the nature of a given Lie algebra $\mathfrak{g}$. Note that the assumption that R commutes with directional derivatives is an important condition although is not enunciated explicitly in most works on the R-matrices.

It is natural to ask when the abstract Lax hierarchy (6) represents a 'real' hierarchy of integrable evolution systems on a suitable function space constituting an infinite-dimensional phase space $\mathcal{U}$. This occurs when we can construct an embedding map $\iota : \mathcal{U} \to \mathfrak{g}$, which induces the differential structure on $\mathfrak{g}$. Note that for ι being an embedding its differential $\iota' : \mathcal{V} \to \mathfrak{g}$ is an injective map, where $\mathcal{V}$ is a space of vector fields on $\mathcal{U}$. In such a case the Lax hierarchy (6) can be pulled back to the original function space by ι'^{-1}. The symmetries from the Lax hierarchy (6) represent compatible evolution systems when the left- and right-hand sides of (6) span the same subspace of $\mathfrak{g}$. So, the Lax element L of $\mathfrak{g}$ has to be chosen in a suitable fashion.

2.3. Simplest R-matrices

The simplest way to obtain a classical R-matrix is to decompose a given Lie algebra into Lie subalgebras. Thus, assume that a Lie algebra $\mathfrak{g}$ can be split into a (vector) direct sum of Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$, i.e.,

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \qquad [\mathfrak{g}_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm \qquad \mathfrak{g}_+ \cap \mathfrak{g}_- = \emptyset.$$

It is important to stress that we do not require that $[\mathfrak{g}_+, \mathfrak{g}_-] = 0$.

Upon denoting the projections onto the subalgebras in question by $P_\pm$, we define a linear map $R : \mathfrak{g} \to \mathfrak{g}$ as

$$R = \tfrac{1}{2}(P_+ - P_-). \tag{12}$$

Using the equality $P_+ + P_- = 1$ (12) can be represented in the following equivalent forms:

$$R = P_+ - \tfrac{1}{2} = \tfrac{1}{2} - P_-. \tag{13}$$

Let $a_\pm := P_\pm(a)$ for $a \in \mathfrak{g}$. Then

$$[a, b]_R = [a_+, b_+] - [a_-, b_-] \quad \Longrightarrow \quad R[a, b]_R = \tfrac{1}{2}[a_+, b_+] + \tfrac{1}{2}[a_-, b_-]$$

and

$$[Ra, Rb] = \tfrac{1}{4}[a_+, b_+] - \tfrac{1}{4}[a_+, b_-] - \tfrac{1}{4}[a_-, b_+] + \tfrac{1}{4}[a_-, b_-].$$

Hence, the map (12) satisfies the Yang–Baxter equation (3) for $\alpha = \frac{1}{4}$ and is a well-defined classical R-matrix. This is the simplest and the most common example of a well-defined R-matrix.

For instance, the Lax hierarchy (6) for the R-matrix (12), following from the decomposition of a Lie algebra into Lie subalgebras, takes the form

$$L_{t_n} = [(X_n(L))_+, L] = -[(X_n(L))_-, L].$$

It can be written in two equivalent ways because (13) holds.

The construction of the majority of known integrable systems within the formalism presented above is based on the classical R-matrices that follow from the double decomposition (12) of Lie algebras into Lie subalgebras. In [64, 84, 88, 94], the authors considered deformations of (12) that preserve the Yang–Baxter equation and originate from a triple decomposition of a given Lie algebra; see also [96] for multiple decompositions of Lie algebras.

2.4. Lie–Poisson structures

Let $\mathfrak{g}^*$ be a (regular) dual of a given Lie algebra $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{K}$ be the usual duality pairing. The co-adjoint action ad^* of $\mathfrak{g}$ on $\mathfrak{g}^*$ is defined through the relation

$$\langle \mathrm{ad}^*_a \eta, b\rangle + \langle \eta, \mathrm{ad}_a b\rangle = 0 \quad \Longleftrightarrow \quad \langle \mathrm{ad}^*_a \eta, b\rangle = -\langle \eta, [a, b]\rangle, \tag{14}$$

where $a, b \in \mathfrak{g}$ and $\eta \in \mathfrak{g}^*$.

Let this time $\iota : \mathcal{U} \to \mathfrak{g}^*$ be the embedding of the original phase space into the dual Lie algebra. Then every functional $F : \mathcal{U} \to \mathbb{K}$ can be extended to a smooth function on $\mathfrak{g}^*$. Therefore, let $\mathcal{C}^\infty(\mathfrak{g}^*)$ be the space of all smooth functions on $\mathfrak{g}^*$ of the form $F \circ \iota^{-1} : \mathfrak{g}^\star \to \mathbb{K}$, where $F \in \mathcal{C}^\infty(\mathcal{U})$. Then the differentials $\mathrm{d}F(\eta)$ of $F(\eta) \in \mathcal{C}^\infty(\mathfrak{g}^*)$ at the point $\eta \in \mathfrak{g}^*$ belong to $\mathfrak{g}$ as they can be evaluated using the relation

$$F(\eta)'[\xi] \equiv \left.\frac{\mathrm{d}F(\eta + \epsilon\xi)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \langle \xi, \mathrm{d}F(\eta)\rangle \qquad \xi \in \mathfrak{g}^* \quad \epsilon \in \mathbb{K}. \tag{15}$$

Moreover, the form of differentials $\mathrm{d}F \in \mathfrak{g}$ has to be such that the duality pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ coincides with the duality map between vector fields and 1-forms on the original infinite-dimensional function phase space $\mathcal{U}$. Indeed,

$$\langle \eta_t, \mathrm{d}F\rangle = \int \sum_{i=0}^{\infty} \frac{\delta F}{\delta u_i}(u_i)_t\, \mathrm{d}x, \tag{16}$$

where $\eta_t \in \mathfrak{g}^*$ is a vector field on $\mathfrak{g}^*$, $F(\eta) \in \mathcal{C}^\infty(\mathfrak{g}^*)$ is a functional depending on the dynamical fields u_i from the phase space $\mathcal{U}$, $\frac{\delta F}{\delta u_i}$ is the variational derivative of F with respect to field variable u_i.

The aim of the considered formalism is the construction of infinite-dimensional field systems in (1+1) and (2+1) dimensions. Thus, the duality map for the abstract algebra $\mathfrak{g}$ must be in agreement with the duality pairing between vector fields and variational differentials from the original functional phase space. Thus, the pairing as well as functionals must be given by appropriate integrals over space coordinates or respective summations in a discrete case.

We also have the relation

$$\langle \xi, \mathrm{d}F'[\eta]\rangle = \langle \eta, \mathrm{d}F'[\xi]\rangle, \tag{17}$$

which is equivalent to the vanishing of the square of the exterior differential, i.e., $\mathrm{d}^2F = 0$.[4]

We also make an additional assumption that the Lie bracket in $\mathfrak{g}$ is such that directional derivative along an arbitrary $\xi \in \mathfrak{g}^\star$ is a derivation of the Lie bracket. This means that the following relation holds:

$$[a, b]'[\xi] = [a'[\xi], b] + [a, b'[\xi]]. \tag{18}$$

Theorem 2.5. *There exists a Poisson bracket on the space of smooth functions on a dual algebra $\mathfrak{g}^*$, which is induced by the Lie bracket on $\mathfrak{g}$. This Poisson bracket is defined as follows:*

$$\{H, F\}(\eta) := \langle \eta, [\mathrm{d}F, \mathrm{d}H]\rangle \qquad \eta \in \mathfrak{g}^* \quad H, F \in \mathcal{C}^\infty(\mathfrak{g}^*). \tag{19}$$

In the case of finite-dimensional smooth manifolds the proof of the above theorem is straightforward as it is enough to consider structure constants. In the infinite-dimensional case the situation is more complex, so we give the 'coordinate-free' proof valid in both cases.

Lemma 2.6. *The differential of (19) is given by*

$$\mathrm{d}\{H, F\} = [\mathrm{d}F, \mathrm{d}H] + \mathrm{d}F'[\mathrm{ad}^*_{\mathrm{d}H}\eta] - \mathrm{d}H'[\mathrm{ad}^*_{\mathrm{d}F}\eta]. \tag{20}$$

Proof. By (15) we find that

$$\begin{aligned}\{H, F\}'[\xi] &= \langle \eta'[\xi], [\mathrm{d}F, \mathrm{d}H]\rangle + \langle \eta, [\mathrm{d}F'[\xi], \mathrm{d}H] + [\mathrm{d}F, \mathrm{d}H'[\xi]]\rangle \\ &\overset{\text{by}(14)}{=} \langle \xi, [\mathrm{d}F, \mathrm{d}H]\rangle + \langle \mathrm{ad}^*_{\mathrm{d}H}\eta, \mathrm{d}F'[\xi]\rangle - \langle \mathrm{ad}^*_{\mathrm{d}F}\eta, \mathrm{d}H'[\xi]\rangle \\ &\overset{\text{by}(17)}{=} \langle \xi, [\mathrm{d}F, \mathrm{d}H] + \mathrm{d}F'[\mathrm{ad}^*_{\mathrm{d}H}\eta] - \mathrm{d}H'[\mathrm{ad}^*_{\mathrm{d}F}\eta]\rangle,\end{aligned}$$

and the result follows. □

[4] The exterior differential can be uniquely defined on $\mathfrak{g}^*$ by means of directional derivative, i.e. $\mathrm{d}\omega(\xi_1, \dots, \xi_{q+1}) = \sum_i (-1)^{i+1}\omega'[\xi_i](\xi_1, \dots, \hat{\xi}_i, \dots, \xi_{q+1})$, where ω is a q-form and $\xi_i \in \mathfrak{g}^*$.

Proof of theorem 2.5. Bilinearity and skew-symmetry of (19) are obvious, so we only have to prove the Jacobi identity:

$$\begin{aligned}\{F,\{G,H\}\}+\text{c.p.} &= \langle\eta,[\mathrm{d}\{G,H\},\mathrm{d}H]+\text{c.p.}\rangle\\ &\overset{\text{by}(20)}{=} \langle\eta,[[\mathrm{d}H,\mathrm{d}G],\mathrm{d}F]+[\mathrm{d}H'[\mathrm{ad}^*_{\mathrm{d}G}\eta],\mathrm{d}F]-[\mathrm{d}G'[\mathrm{ad}^*_{\mathrm{d}H}\eta],\mathrm{d}F]+\text{c.p.}\rangle\\ &\overset{\text{by}(14)}{=} \langle\eta,[[\mathrm{d}H,\mathrm{d}G],\mathrm{d}F]\rangle+\langle\mathrm{ad}^*_{\mathrm{d}F}\eta,\mathrm{d}H'[\mathrm{ad}^*_{\mathrm{d}G}\eta]\rangle-\langle\mathrm{ad}^*_{\mathrm{d}F}\eta,\mathrm{d}G'[\mathrm{ad}^*_{\mathrm{d}H}\eta]\rangle+\text{c.p.}\\ &\overset{\text{by c.p.}}{=} \langle\eta,[[\mathrm{d}H,\mathrm{d}G],\mathrm{d}F]\rangle+\langle\mathrm{ad}^*_{\mathrm{d}F}\eta,\mathrm{d}H'[\mathrm{ad}^*_{\mathrm{d}G}\eta]\rangle-\langle\mathrm{ad}^*_{\mathrm{d}G}\eta,\mathrm{d}H'[\mathrm{ad}^*_{\mathrm{d}F}\eta]\rangle+\text{c.p.}\\ &\overset{\text{by}(17)}{=} \langle\eta,[[\mathrm{d}H,\mathrm{d}G],\mathrm{d}F]+\text{c.p.}\rangle=0,\end{aligned}$$

where the last equality follows from the Jacobi identity for $[\cdot,\cdot]$. □

Bracket (19) is called a (natural) *Lie–Poisson bracket* and was originally discovered by Sophus Lie. Its modern formulation is due to Berezin [7] as well as Kirillov and Kostant [47].

Now assume that we have an additional Lie bracket (1) on $\mathfrak{g}$ defined through the classical R-matrix such that (10) is valid. Then (1) satisfies condition (18). As a result, there is another well-defined (by theorem 2.5) Lie–Poisson bracket on the space of scalar fields $\mathcal{C}^\infty(\mathfrak{g}^*)$:

$$\{H,F\}_R(\eta):=\langle\eta,[\mathrm{d}F,\mathrm{d}H]_R\rangle\qquad \eta\in\mathfrak{g}^*\quad H,F\in\mathcal{C}^\infty(\mathfrak{g}^*).\tag{21}$$

Using (14) we find that the associated Poisson operators at $\eta\in\mathfrak{g}^*$, the one for the natural Lie–Poisson bracket (19) and the second one, (21), have the form

$$\{H,F\}=\langle\pi\,\mathrm{d}H,\mathrm{d}F\rangle\quad\Longleftrightarrow\quad \pi:\mathrm{d}H\mapsto\mathrm{ad}^*_{\mathrm{d}H}\eta$$
$$\{H,F\}_R=\langle\pi_R\,\mathrm{d}H,\mathrm{d}F\rangle\quad\Longleftrightarrow\quad \pi_R:\mathrm{d}H\mapsto\mathrm{ad}^*_{R\mathrm{d}H}\eta+R^*\mathrm{ad}^*_{\mathrm{d}H}\eta,$$

where the adjoint of R is defined by the relation $\langle R^*\eta,a\rangle=\langle\eta,Ra\rangle$, where $\eta\in\mathfrak{g}^*$ and $a\in\mathfrak{g}$.

The following theorem constitutes the essence of the classical R-matrix formalism.

Theorem 2.7 [81]. *The Casimir functions $C_n\in\mathcal{C}^\infty(\mathfrak{g}^*)$ of the natural Lie–Poisson bracket (19) are in involution with respect to the Lie–Poisson bracket (21) induced by (1). Moreover, C_n generate a hierarchy of vector fields on $\mathfrak{g}^*$:*

$$\eta_{t_n}=\pi_R\,\mathrm{d}C_n(\eta)=\mathrm{ad}^*_{R\mathrm{d}C_n}\eta\qquad\eta\in\mathfrak{g}^*.\tag{22}$$

The evolution systems (22) pairwise commute, i.e., $(\eta_{t_m})_{t_n}=(\eta_{t_n})_{t_m}$, and are Hamiltonian with respect to (21). Moreover, any equation from (22) admits all Casimir functions C_n of (19) as integrals of motion.

Proof. The Casimir functions C_n of the natural Lie–Poisson bracket (19) satisfy the following condition:

$$\forall F\in\mathcal{C}^\infty(\mathfrak{g}^*)\quad\{F,C_n\}=0\quad\Longleftrightarrow\quad\mathrm{ad}^*_{\mathrm{d}C_n}\eta=0,$$

that is, their differentials are ad^*-invariant. Hence, they are in involution with respect to the Lie–Poisson bracket (21), i.e., $\{C_n,C_m\}_R=0$. Now, as $\pi_R d$ is a Lie algebra homomorphism from the Poisson algebra of smooth functions on $\mathfrak{g}^*$ to the Lie algebra of vector fields on $\mathfrak{g}^*$, commutativity of Hamiltonian vector fields (22), with the Casimir functions as Hamiltonians, readily follows. □

In fact, when the R-matrix follows from the decomposition of an Lie algebra into a sum of Lie subalgebras, i.e. R is given by (12), then theorem 2.7 can be considered as a generalization of the Kostant–Symes theorem [50, 92].

The construction of Casimir functions C_n and related dynamical systems (22) on the dual Lie algebra $\mathfrak{g}^*$ is, in contrast with (6), often inconvenient and impractical. Thus, a formulation of a similar theory on $\mathfrak{g}$ instead on $\mathfrak{g}^*$ is often justified. This can be done when one can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of a suitable scalar product.

2.5. Ad-invariant scalar products

We restrict our further considerations to the Lie algebras $\mathfrak{g}$ for which their duals $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$ through a duality map. So, we assume the existence of a bilinear scalar product

$$(\cdot,\cdot)_{\mathfrak{g}} : \mathfrak{g}\times\mathfrak{g}\to\mathbb{K} \tag{23}$$

on $\mathfrak{g}$, such that it is symmetric: $(a,b)_{\mathfrak{g}} = (b,a)_{\mathfrak{g}}$, and non-degenerate, that is, $a = 0$ is the only element of $\mathfrak{g}$ that satisfies $(a,b)_{\mathfrak{g}} = 0$ for all $b\in\mathfrak{g}$. Then we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ ($\mathfrak{g}^*\cong\mathfrak{g}$) by setting $\langle\eta,b\rangle = (c,b)_{\mathfrak{g}}, \forall\, b\in\mathfrak{g}$, where $\eta\in\mathfrak{g}^*$ is identified with $c\in\mathfrak{g}$.

We also make an additional assumption that the symmetric product (23) is ad-invariant, i.e.,

$$([a,c],b)_{\mathfrak{g}} + (c,[a,b])_{\mathfrak{g}} = 0. \tag{24}$$

This is a counterpart of relation (14). Thus, if $\eta\in\mathfrak{g}^*$ is identified with $c\in\mathfrak{g}$ we have $\langle \mathrm{ad}^*_a\eta, b\rangle = ([a,c],b)_{\mathfrak{g}}$ and one identifies $\mathrm{ad}^*_a\eta\in\mathfrak{g}^*$ with $\mathrm{ad}_a c\in\mathfrak{g}$.

In fact, under the above assumptions and by virtue of the scheme presented in the previous section all equations from the hierarchy (6) in principle are Hamiltonian. Since $\mathfrak{g}^*\cong\mathfrak{g}$, the Lie–Poisson brackets (19) and (21) on the space of scalar fields $\mathcal{C}^\infty(\mathfrak{g}\cong\mathfrak{g}^*)$ at $L\in\mathfrak{g}$ take the form

$$\begin{aligned} &\{H,F\} = (L,[\mathrm{d}F,\mathrm{d}H])_{\mathfrak{g}} = (\mathrm{d}F,\pi\,\mathrm{d}H)_{\mathfrak{g}} \quad\Longleftrightarrow\quad \pi\,\mathrm{d}H = [\mathrm{d}H,L]\\ &\{H,F\}_R = (L,[\mathrm{d}F,\mathrm{d}H]_R)_{\mathfrak{g}} = (\mathrm{d}F,\pi_R\,\mathrm{d}H)_{\mathfrak{g}} \iff \pi_R\,\mathrm{d}H = [R\,\mathrm{d}H,L] + R^*[\mathrm{d}H,L], \end{aligned} \tag{25}$$

where now R^* is defined by the relation $(R^*a,b)_{\mathfrak{g}} = (a,Rb)_{\mathfrak{g}}$.

Differentials of the Casimir functions $\mathcal{C}_n(L)\in\mathcal{C}^\infty(\mathfrak{g})$ of the natural Lie–Poisson bracket are invariants of the Lie bracket, i.e., $[\mathrm{d}C_n(L),L] = 0$. Obviously, the Casimir functions are still in involution with respect to the second Lie–Poisson bracket defined by R and generate pairwise commuting Hamiltonian vector fields of the form

$$L_{t_n} = \pi_R\,\mathrm{d}C_n(L) = [R\,\mathrm{d}C_n, L]. \tag{26}$$

Note that the Lax hierarchy (6) coincides with (26) for $X_n = \mathrm{d}C_n$. Moreover, it follows that if there exists a symmetric, non-degenerate and ad-invariant product on $\mathfrak{g}$ then the Yang–Baxter equation (3) is not a necessary condition for the commutativity of vector fields from the Lax hierarchy (6). However, if (3) is not satisfied then the zero-curvature equations (9) will not be automatically satisfied as well.

The simplest way to define an appropriate scalar product on a Lie algebra $\mathfrak{g}$ is to use a trace form $\mathrm{Tr}:\mathfrak{g}\to\mathbb{K}$ such that the scalar product

$$(a,b)_{\mathfrak{g}} := \mathrm{Tr}(ab) \qquad a,b\in\mathfrak{g} \tag{27}$$

is nondegenerate. In this case the symmetry of (27) entails that

$$\mathrm{Tr}(ab) = \mathrm{Tr}(ba). \tag{28}$$

Lemma 2.8. *Let* $\mathrm{Tr}:\mathfrak{g}\to\mathbb{K}$ *be a trace form defining a symmetric and nondegenerate scalar product (27) such that the trace of the Lie bracket vanishes, i.e.* $\mathrm{Tr}[a,b] = 0$ *for all* $a,b\in\mathfrak{g}$*. Then condition (4) for the Lie bracket to be a derivation with respect to the multiplication is a sufficient condition for (27) to be ad-invariant.*

Moreover, if the Lie bracket in $\mathfrak{g}$ *is given by the commutator,* $[a,b] = ab - ba$*, then the ad-invariance follows from the associativity of the multiplication in* $\mathfrak{g}$*.*

Proof. The first part of the lemma is immediate, as

$$([a,c],b)_{\mathfrak{g}} + (c,[a,b])_{\mathfrak{g}} = \mathrm{Tr}([a,c]b + c[a,b]) = \mathrm{Tr}[a,cb] = 0,$$

where we used the assumptions from the proposition. The second statement of the lemma follows immediately from the assumption and (28). □

Under the assumption that we have an algebra $\mathfrak{g}$ such that a Lie bracket is a derivation of a multiplication (4) and $\mathfrak{g}$ is endowed with a trace form inducing a nondegenerate ad-invariant scalar product (27), the most natural Casimir functions $\mathcal{C}_n(L) \in \mathcal{C}^\infty(\mathfrak{g})$ of the Lie–Poisson bracket (19) are given by the traces of powers of L, i.e.,

$$C_n(L) = \frac{1}{n+1}\mathrm{Tr}(L^{n+1}) \qquad \Longleftrightarrow \qquad \mathrm{d}C_n = L^n \quad n \neq -1. \tag{29}$$

The associated differentials are found from expression (15), which can be now reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}F(L) = (L_t, \mathrm{d}F)_{\mathfrak{g}} = \mathrm{Tr}(L_t\,\mathrm{d}F) \qquad L \in \mathfrak{g}, \tag{30}$$

where t is an evolution parameter associated with a vector field L_t on $\mathfrak{g}$.

2.6. Hamiltonian structures on Poisson algebras

Definition 2.9. *Let $\mathcal{A}$ be a commutative, associative algebra with unit 1. If there is a Lie bracket on $\mathcal{A}$ such that for each element $a \in \mathcal{A}$ the operator $\mathrm{ad}_a : b \mapsto \{a,b\}$ is a derivation of the multiplication, i.e. $\{a,bc\} = \{a,b\}c + b\{a,c\}$, then $(\mathcal{A},\{\cdot,\cdot\})$ is called a Poisson algebra and the bracket $\{\cdot,\cdot\}$ is a Poisson bracket.*

Thus, the Poisson algebras are Lie algebras with the Lie bracket $[\cdot,\cdot] := \{\cdot,\cdot\}$ endowed with an additional structure. Of course, we should not confuse the above bracket with the Poisson brackets in the algebra of scalar fields. It will follow easily from the context which bracket is used. In the case of the Poisson algebra $\mathcal{A}$ a classical R-matrix defines the second Lie product on $\mathcal{A}$ but not the Poisson bracket; in general, this would not be possible.

Theorem 2.10 [55]**.** *Let $\mathcal{A}$ be a Poisson algebra with the Poisson bracket $\{\cdot,\cdot\}$ and a non-degenerate ad-invariant scalar product $(\cdot,\cdot)_{\mathcal{A}}$ such that the operation of multiplication is symmetric with respect to the latter, i.e., $(ab,c)_{\mathcal{A}} = (a,bc)_{\mathcal{A}}$ for all $a,b,c \in \mathcal{A}$. Assume that R is a classical R-matrix such that (10) holds.*

Then for any integer $n \geqslant 0$ the formula

$$\{H,F\}_n = (L, \{R(L^n\,\mathrm{d}F), \mathrm{d}H\} + \{\mathrm{d}F, R(L^n\,\mathrm{d}H)\})_{\mathcal{A}}, \tag{31}$$

where H, F are smooth functions on $\mathcal{A}$, defines a Poisson structure on $\mathcal{A}$. Moreover, all brackets $\{\cdot,\cdot\}_n$ are compatible.

For the proof we send reader to the original publication [55].

An important property that classical R-matrices commute with differentials of smooth maps from $\mathcal{A}$ to $\mathcal{A}$, or equivalently satisfy (10), is used in the proof of theorem 4.2 of [55], although it is not explicitly stated there. In fact, the existence of scalar product being symmetric with respect to the multiplication, $(ab,c) = (a,bc)$, entails existence of a trace form on $\mathcal{A}$. Setting $c = 1$ we have $(ab,1) = (a,b)$. Thus, the trace can be defined as $\mathrm{Tr}(a) := (a,1) = (1,a)$.

The Poisson operators π_n related to the Poisson brackets (31) such that $\{H,F\}_n = (\mathrm{d}F, \pi_n\,\mathrm{d}H)$, are given by the following Poisson maps:

$$\pi_n : \mathrm{d}H \mapsto \{R(L^n\,\mathrm{d}H), L\} + L^n R^*(\{\mathrm{d}H, L\}) \qquad n \geqslant 0. \tag{32}$$

Note that the bracket (31) with $n = 0$ is just the Lie–Poisson bracket with respect to the second Lie bracket on $\mathcal{A}$ defined by a classical R-matrix. Referring to the dependence on L, the Poisson maps (32) are called linear for $n = 0$, quadratic for $n = 1$ and cubic for $n = 2$, respectively. The Casimir functions $C(L)$ of the natural Lie–Poisson bracket are in involution with respect to all Poisson brackets (32) and generate pairwise commuting Hamiltonian vector fields of the form

$$L_t = \pi_n\, \mathrm{d}C = \{R(L^n\, \mathrm{d}C), L\} \qquad L \in \mathcal{A}.$$

Taking the most natural Casimir functions (29), defined by traces of powers of L, for the Hamiltonians, one finds a hierarchy of evolution equations which are multi-Hamiltonian dynamical systems:

$$L_{t_n} = \{R\, \mathrm{d}C_n, L\} = \pi_0\, \mathrm{d}C_n = \pi_1\, \mathrm{d}C_{n-1} = \cdots = \pi_l\, \mathrm{d}C_{n-l} = \cdots, \tag{33}$$

where C_n are such that $\mathrm{d}C_n = L\, \mathrm{d}C_{n-1}$. For any R-matrix any two evolution equations in the hierarchy (33) commute because of involutivity of the Casimir functions C_n. Each equation admits all the Casimir functions as conserved quantities in involution. In this sense, we will consider (33) as a hierarchy of integrable evolution equations. The most natural choice for the Casimir functions is the traces of the power functions (29).

2.7. *Hamiltonian structures on noncommutative algebras*

In this section, in contrast with the previous one, we will consider a noncommutative associative algebra $\mathfrak{g}$, with unity, for which the Lie structure is defined as a commutator, i.e., $[a, b] := ab - ba$, where $a, b \in \mathfrak{g}$. Such a Lie bracket automatically satisfies the required Leibniz rule (4). We further assume existence of nondegenerate, symmetric and ad-invariant scalar product on $\mathfrak{g}$. Let R be a classical R-matrix such that (10) is satisfied.

In this case, the situation is more involved and only three explicit forms of Poisson brackets on the space of smooth functions $\mathcal{C}^\infty(\mathfrak{g})$ defined by related Poisson tensors are known from the literature:

$$\{H, F\}_n = (\mathrm{d}F, \pi_n\, \mathrm{d}H)_{\mathfrak{g}} \qquad n = 0, 1, 2.$$

These Poisson brackets (or associated tensors) are called linear, quadratic and cubic brackets (resp. tensors) for $n = 0, 1, 2$, respectively.

The linear one is simply the Lie–Poisson bracket, with respect to the second Lie structure on $\mathfrak{g}$ defined by classical R-matrix, with the Poisson tensor (25)

$$\pi_0\, \mathrm{d}H = [R\, \mathrm{d}H, L] + R^*[\mathrm{d}H, L], \tag{34}$$

for which there is no need for additional assumptions.

In our further considerations we have to assume that the scalar product is symmetric with respect to the operation of multiplication, $(ab, c) = (a, bc)$. Note that this property implies that the scalar product is automatically ad-invariant with respect to the Lie bracket defined by the commutator (see lemma 2.8).

The quadratic case is more delicate. A quadratic tensor [91]

$$\pi_1\, \mathrm{d}H = A_1(L\, \mathrm{d}H)L - LA_2(\mathrm{d}HL) + S(\mathrm{d}HL)L - LS^*(L\, \mathrm{d}H) \tag{35}$$

defines a Poisson tensor if the linear maps $A_{1,2} : \mathfrak{g} \to \mathfrak{g}$ are skew-symmetric, $A^*_{1,2} = -A_{1,2}$, satisfies YB(α) (3) for $\alpha \neq 0$ and the linear map $S : \mathfrak{g} \to \mathfrak{g}$ with its adjoint S^* satisfies

$$S([A_2a, b] + [a, A_2b]) = [Sa, Sb], \qquad S^*([A_1a, b] + [a, A_1b]) = [S^*a, S^*b]. \tag{36}$$

In the special case when

$$\widetilde{R} := \tfrac{1}{2}(R - R^*) \tag{37}$$

satisfies YB(α), for the same α as R, under the substitution $A_1 = A_2 = R - R^*$ and $S = S^* = R + R^*$ the quadratic Poisson operator (35) reduces to [56, 66]

$$\pi_1\, \mathrm{d}H = [R[\mathrm{d}H, L]_+, L] + LR^*[\mathrm{d}H, L] + R^*([\mathrm{d}H, L])L, \tag{38}$$

where $[a, b]_+ := ab + ba$ and conditions (36) are equivalent to YB(α) for R and $\widetilde{R}$. In particular, when $R^* = -R$, these conditions are automatically satisfied as in this case $\widetilde{R} = R$.

Another special case occurs when the maps $A_{1,2}$ and S originate from the decomposition of a given classical R-matrix satisfying YB(α) for $\alpha \neq 0$

$$R = \tfrac{1}{2}(A_1 + S) = \tfrac{1}{2}(A_2 + S^*),$$

where $A_{1,2}$ are skew-symmetric. Then, conditions (36) imply that both A_1 and A_2 satisfy YB(α) for the same value of α as R [64]. Hence, in this case we only have to check conditions (36) for (35) to be a Poisson operator. The latter now takes the form

$$\pi_1 : \mathrm{d}H \mapsto 2R(L\,\mathrm{d}H)L - 2LR(\mathrm{d}HL) + S([\mathrm{d}H, L])L + LS^*[\mathrm{d}H, L]. \tag{39}$$

Finally, the cubic tensor π_2 takes the simple form [66]

$$\pi_2 : \mathrm{d}H \mapsto [R(L\,\mathrm{d}HL), L] + LR^*([\mathrm{d}H, L])L$$

and is Poisson without any further additional assumptions.

Once again, taking the Casimir functions (29) defined by the traces of powers of L for the Hamiltonians yields a hierarchy of evolution equations which are tri-Hamiltonian dynamical systems,

$$L_{t_n} = [R\,\mathrm{d}C_n, L] = \pi_0\, \mathrm{d}C_n = \pi_1\, \mathrm{d}C_{n-1} = \pi_2\, \mathrm{d}C_{n-2}, \tag{40}$$

where C_n are such that $\mathrm{d}C_n = L\,\mathrm{d}C_{n-1}$. We assumed that π_2 in (40) is given by (38) or (39). In the first case all three Poisson tensors in (40) are automatically compatible. In the second case, this has to be checked separately.

2.8. *Central extension approach*

Let $\mathfrak{g}$ be a Lie algebra with the Lie bracket $[\cdot,\cdot]$. Consider its extension $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{K}$ with the Lie bracket given by

$$[(a, \alpha), (b, \beta)] \equiv \widehat{\mathrm{ad}}_{(a,\alpha)}(b, \beta) := ([a, b], \omega(a, b)) \qquad a, b \in \mathfrak{g} \quad \alpha, \beta \in \mathbb{K}, \tag{41}$$

where $\widehat{\mathrm{ad}}$ is the associated adjoint action. It is readily seen that (41) is a well-defined Lie bracket if and only if ω is a 2-cocycle.

Definition 2.11. *A 2-cocycle on $\mathfrak{g}$ is a bilinear map $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ such that*

(i) it is skew-symmetric: $\omega(a, b) = -\omega(b, a)$,
(ii) and it satisfies the cyclic condition:

$$\omega([a, b], c) + \omega([c, a], b) + \omega([b, c], a) = 0, \tag{42}$$

where $a, b, c \in \mathfrak{g}$.

Note that $(0, \alpha) \in \widehat{\mathfrak{g}}$ commute with respect to (41) with all other elements from $\widehat{\mathfrak{g}}$ and hence $\mathfrak{g}$ can be identified with $\mathfrak{g} \oplus \alpha$ for fixed $\alpha \in \mathbb{K}$. The value α is often called a charge. In fact $\mathfrak{g} \cong \widehat{\mathfrak{g}}/\mathfrak{c}$, where $\mathfrak{c} = \{(0, \alpha) \in \widehat{\mathfrak{g}} : \alpha \in \mathbb{K}\}$ is in the center of $\widehat{\mathfrak{g}}$, and thus the Lie algebra $\widehat{\mathfrak{g}}$ is called a *central extension* of $\mathfrak{g}$.

Assume now (for simplicity) that $\mathfrak{g}$ can be identified with $\mathfrak{g}^*$ through a non-degenerate symmetric scalar product (23). Then this product can be extended to the algebra $\widehat{\mathfrak{g}}$ in the following fashion:

$$((a, \alpha), (b, \beta))_{\widehat{\mathfrak{g}}} := (a, b)_{\mathfrak{g}} + \alpha\beta \qquad a, b \in \mathfrak{g} \quad \alpha, \beta \in \mathbb{K}. \tag{43}$$

Of course, (43) is symmetric and an important fact is that it preserves non-degeneracy. Thus, $\widehat{\mathfrak{g}}^*$ can be identified with $\widehat{\mathfrak{g}}$.

Usually 2-cocycles are defined through the scalar product on $\mathfrak{g}$ and a linear map $\phi : \mathfrak{g} \to \mathfrak{g}$ such that

$$\omega(a, b) = (a, \phi(b))_{\mathfrak{g}}. \tag{44}$$

If the linear map $\phi : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric, i.e., $\phi^* = -\phi$, and the following condition

$$\phi([a, b]) = \mathrm{ad}^*_a \phi(b) - \mathrm{ad}^*_b \phi(a) \tag{45}$$

is satisfied, then it is called a *1-cocycle*.

Proposition 2.12. *The bilinear form given by (44) is a 2-cocycle if and only if ϕ is a 1-cocycle. Moreover, if the symmetric product on $\mathfrak{g}$ is ad-invariant (24) then (44) is a 2-cocycle if and only if the skew-symmetric ϕ is a derivation of the Lie bracket $[\cdot, \cdot]$ in $\mathfrak{g}$, i.e.,*

$$\phi([a, b]) = [\phi(a), b] + [a, \phi(b)] \tag{46}$$

holds.

Proof. It is clear that (44) is skew-symmetric if and only if $\phi^* = -\phi$. The cyclic condition (42) for skew-symmetric ϕ has the form

$$\begin{aligned}\omega([a, b], c) + \text{c.p.} &= -(\phi([a, b]), c)_{\mathfrak{g}} + ([b, c], \phi(a))_{\mathfrak{g}} + ([c, a], \phi(b))_{\mathfrak{g}}\\ &= -(\phi([a, b]), c)_{\mathfrak{g}} - (\mathrm{ad}^*_b \phi(a), c)_{\mathfrak{g}} + (\mathrm{ad}^*_a \phi(b), c)_{\mathfrak{g}} = 0,\end{aligned}$$

where we used definition (14) of coadjoint action. Now, since the symmetric product on $\mathfrak{g}$ is non-degenerate, the cyclic condition is equivalent to (45). For the ad-invariant symmetric product (46) follows from (45) since ad^* is in this case identified with ad. □

The adjoint action (41) does not depend on α, thus in fact it defines adjoint action of $\mathfrak{g}$ on $\widehat{\mathfrak{g}}$. Hence, we can omit dependence on the charge and write $\widehat{\mathrm{ad}}_a \equiv \widehat{\mathrm{ad}}_{(a,\alpha)}$. When a given 2-cocycle is defined by means of a 1-cocycle, i.e., (44) holds, then we can write explicitly the coadjoint action $\widehat{\mathrm{ad}}^*$, since

$$\begin{aligned}(\widehat{\mathrm{ad}}^*_b(a, \alpha), (c, \gamma))_{\widehat{\mathfrak{g}}} &:= -((a, \alpha), \widehat{\mathrm{ad}}_b(c, \gamma))_{\widehat{\mathfrak{g}}} = -(a, \mathrm{ad}_b c)_{\mathfrak{g}} - \alpha(b, \phi(c))_{\mathfrak{g}}\\ &= (\mathrm{ad}^*_b a + \alpha\phi(b), c)_{\mathfrak{g}} = ((\mathrm{ad}^*_b a + \alpha\phi(b), 0), (c, \gamma))_{\widehat{\mathfrak{g}}}.\end{aligned}$$

Hence, we can restrict $\widehat{\mathrm{ad}}^*$ to $\mathfrak{g}$ and define the central extension of coadjoint action of $\mathfrak{g}^*$ on $\mathfrak{g}$, i.e.

$$\widehat{\mathrm{ad}}^*_b(\cdot) := \mathrm{ad}^*_b(\cdot) + \alpha\phi(b)$$

for every $b \in \mathfrak{g}$, where the charge $\alpha \in \mathbb{K}$ is now treated as a parameter.

According to theorem 2.5 the Lie bracket (41) on $\widehat{\mathfrak{g}}$ defines a Lie–Poisson bracket on the space of smooth functions on $\widehat{\mathfrak{g}}^* \cong \widehat{\mathfrak{g}}$, i.e., on $\mathcal{C}^\infty(\widehat{\mathfrak{g}})$. This Poisson bracket can be restricted to $\mathcal{C}^\infty(\mathfrak{g})$ considered as a subspace of $\mathcal{C}^\infty(\widehat{\mathfrak{g}})$. Hence, at a point $(L, \alpha) \in \widehat{\mathfrak{g}}$ we have

$$\{H, F\}(L) := ((L, \alpha), [(\mathrm{d}F, 0), (\mathrm{d}H, 0)])_{\widehat{\mathfrak{g}}} = (L, [\mathrm{d}F, \mathrm{d}H])_{\mathfrak{g}} + \alpha\omega(\mathrm{d}F, \mathrm{d}H), \tag{47}$$

where $H, F \in \mathcal{C}^\infty(\mathfrak{g})$ and $\alpha \in \mathbb{K}$. The Poisson bracket (47) is a central extension of the natural Lie–Poisson bracket on $\mathcal{C}^\infty(\mathfrak{g})$ generated by the Lie algebra structure on $\mathfrak{g}$. When the 2-cocycle is given by (44), then the associated Poisson tensor π such that $\{H, F\} = (\mathrm{d}F, \pi\, \mathrm{d}H)_{\mathfrak{g}}$ has the form $\pi\, \mathrm{d}H = \widehat{\mathrm{ad}}^*_{\mathrm{d}H} L \equiv \mathrm{ad}^*_{\mathrm{d}H} L + \alpha\phi(\mathrm{d}H)$.

The second Lie bracket on $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{K}$, being an extension of (1), is defined by

$$[(a, \alpha), (b, \beta)]_R := ([a, b]_R, \omega_R(a, b)) \qquad a, b \in \mathfrak{g} \quad \alpha, \beta \in \mathbb{K}, \tag{48}$$

where

$$\omega_R(a, b) = \omega(Ra, b) + \omega(a, Rb), \tag{49}$$

and ω is a 2-cocycle from (41). Clearly, (48) is a well-defined Lie bracket on $\widehat{\mathfrak{g}}$ if and only if $R : \mathfrak{g} \to \mathfrak{g}$ is a classical R-matrix and (49) is a 2-cocycle with respect to the second Lie bracket on $\mathfrak{g}$ (1) induced by R.

Proposition 2.13. *A sufficient condition on R for (49) to be a 2-cocycle with respect to (1) is the Yang–Baxter equation (3).*

Proof. Skew-symmetry of (49) is obvious. Hence, it suffices to verify the cyclic condition for the 2-cocycles. Thus, from the Yang–Baxter equation (3) we have that

$$\begin{aligned}\omega_R([a, b]_R, c) + \text{c.p.} &= \omega(R[a, b]_R, c) + \omega([a, b]_R, Rc) + \text{c.p.}\\ &= \omega([Ra, Rb], c) + \alpha\omega([a, b], c) + \omega([Ra, b], Rc) + \omega([a, Rb], Rc) + \text{c.p.}\\ &= \omega([Ra, Rb], c) + \omega([Rb, c], Ra) + \omega([c, Ra], Rb) + \text{c.p.} = 0,\end{aligned}$$

where the last two equalities hold because ω is a 2-cocycle. □

Hence, the second Lie–Poisson bracket on $\mathcal{C}^\infty(\mathfrak{g})$ at a point $(L, \alpha) \in \widehat{\mathfrak{g}}$ has the form

$$\begin{aligned}\{H, F\}_R(L) &:= ((L, \alpha), [(\mathrm{d}F, 0), (\mathrm{d}H, 0)]_R)_{\widehat{\mathfrak{g}}}\\ &= (L, [\mathrm{d}F, \mathrm{d}H]_R)_{\mathfrak{g}} + \alpha\omega_R(\mathrm{d}F, \mathrm{d}H),\end{aligned} \tag{50}$$

where $H, F \in \mathcal{C}^\infty(\mathfrak{g})$ and $\alpha \in \mathbb{K}$. When a 2-cocycle is given by (44), then the associated Poisson tensor π_R such that $\{H, F\}_R = (\mathrm{d}F, \pi_R\, \mathrm{d}H)_{\mathfrak{g}}$ has the form

$$\pi_R\, \mathrm{d}H = \widehat{\mathrm{ad}}^*_{R\,\mathrm{d}H} L + R^*\widehat{\mathrm{ad}}^*_{\mathrm{d}H} L \equiv \mathrm{ad}^*_{R\,\mathrm{d}H} L + R^*\mathrm{ad}^*_{\mathrm{d}H} L + \alpha\phi(R\,\mathrm{d}H) + \alpha R^*\phi(\mathrm{d}H). \tag{51}$$

Note that the higher order Poisson tensors from sections 2.6 and 2.7 do not survive the procedure of central extension. We have the following straightforward extension of theorem 2.7.

Theorem 2.14. *The Casimir functionals C_n of (47), i.e., $C_n \in \mathcal{C}^\infty(\mathfrak{g})$ such that*

$$\pi\, \mathrm{d}C_n = \mathrm{ad}^*_{dC_n} L + \alpha\phi(\mathrm{d}C_n) = 0, \tag{52}$$

are in involution with respect to (50) and hence generate the following hierarchy of mutually commuting Hamiltonian evolution equations on $\mathfrak{g}$:

$$L_{t_n} = \pi_R\, \mathrm{d}C_n = \mathrm{ad}^*_{R\,\mathrm{d}C_n} L + \alpha\phi(R\,\mathrm{d}C_n). \tag{53}$$

Let us consider an important special case of (44), when ϕ is a derivation with respect to additional continuous space coordinate. Let $\mathfrak{g}$ be a Lie algebra $\mathfrak{g}$ with a non-degenerate symmetric ad-invariant scalar product (27) defined by means of a trace form tr on $\mathfrak{g}$. Assume now that $\mathfrak{g}$ depends in a nontrivial fashion on an additional continuous parameter $y \in \mathbb{S}^1$, which naturally generates the corresponding current operator algebra

$$\widetilde{\mathfrak{g}} = \mathcal{C}^\infty(\mathbb{S}^1, \mathfrak{g}) \tag{54}$$

of smooth maps from $\mathbb{S}^1$ to $\mathfrak{g}$. On the current algebra $\widetilde{\mathfrak{g}}$ we define the following modified trace form $\mathrm{Tr} : \widetilde{\mathfrak{g}} \to \mathbb{K}$, such that $\mathrm{Tr}(a) := \int_{\mathbb{S}^1} \mathrm{tr}(a)\, \mathrm{d}y$, where $a \in \widetilde{\mathfrak{g}}$. Then the scalar product reads

$$(a, b)_{\widetilde{\mathfrak{g}}} := \mathrm{Tr}(ab) = \int_{\mathbb{S}^1} \mathrm{tr}(ab)\, \mathrm{d}y \qquad a, b \in \widetilde{\mathfrak{g}}. \tag{55}$$

Thus, assuming that the derivative with respect to y is a derivation of the Lie bracket in $\widetilde{\mathfrak{g}}$, i.e., (46) with $\phi = \partial_y$ holds, we can define the so-called Maurer–Cartan 2-cocycle

$$\omega(a,b) = (a, \partial_y b)_{\widetilde{\mathfrak{g}}} \equiv \int_{\mathbb{S}^1} \operatorname{tr}(a\partial_y b)\,\mathrm{d}y \qquad a, b \in \widetilde{\mathfrak{g}}. \tag{56}$$

In this case, the Casimir functions of the natural Lie–Poisson bracket on the centrally extended Lie algebra satisfy (52) in the form of the so-called Novikov–Lax equation

$$[\mathrm{d}C_n, L] + \alpha\partial_y(\mathrm{d}C_n) = 0. \tag{57}$$

This follows from the invariance of the scalar product, as in this case $\mathrm{ad}^* \cong \mathrm{ad}$. The differentials of Casimirs $\mathrm{d}C_n$ generate the following Lax hierarchy (53):

$$L_{t_n} = [R\,\mathrm{d}C_n, L] + \alpha\partial_y(R\,\mathrm{d}C_n) = \pi_R\,\mathrm{d}C_n, \tag{58}$$

where the Poisson tensor (51) takes the form

$$\pi_R\,\mathrm{d}H = [R\,\mathrm{d}H, L] + R^*[\mathrm{d}H, L] + \alpha\partial_y(R\,\mathrm{d}H) + \alpha R^*\partial_y(\mathrm{d}H). \tag{59}$$

2.9. *Dirac reduction and homotopy formula*

It often happens that we need to restrict the dynamics under study to a submanifold defined via some constraints. In such a case, a question arises of whether and how one can reduce the Poisson tensors.

Assume that the (linear) phase space $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$ is spanned by $\mathbf{u}_1 \in \mathcal{U}_1$ and $\mathbf{u}_2 \in \mathcal{U}_2$, i.e. $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)^{\mathrm{T}}$. We will only consider the simplest case of the Dirac reduction given by the constraint $\mathbf{u}_2 = \mathbf{c}$, where $\mathbf{c} \in \mathcal{U}_2$ is an arbitrary constant. In many cases considering such constraint is sufficient. Besides, more complicated constraints can always be reduced by change of dependent variables to several constraints of the above type.

The Hamiltonian system with the Poisson tensor before reduction has the form

$$\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}_t = \begin{pmatrix} \pi_{11}(\mathbf{u}_1,\mathbf{u}_2) & \pi_{12}(\mathbf{u}_1,\mathbf{u}_2) \\ \pi_{21}(\mathbf{u}_1,\mathbf{u}_2) & \pi_{22}(\mathbf{u}_1,\mathbf{u}_2) \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \mathbf{u}_1} \\ \frac{\delta H}{\delta \mathbf{u}_2} \end{pmatrix},$$

where we assume that π_{22} is nondegenerate and hence invertible. Taking the constraint $\mathbf{u}_2 = \mathbf{c}$ into consideration we find that

$$0 = (\mathbf{u}_2)_t|_{\mathbf{u}_2=\mathbf{c}} = \pi_{21}(\mathbf{u}_1, \mathbf{c})\frac{\delta H}{\delta \mathbf{u}_1} + \pi_{22}(\mathbf{u}_1, \mathbf{c})\frac{\delta H}{\delta \mathbf{u}_2}.$$

Thus, we can express $\frac{\delta H}{\delta \mathbf{u}_2}$ in the terms of $\frac{\delta H}{\delta \mathbf{u}_1}$ and put it into

$$(\mathbf{u}_1)_t = \pi_{21}(\mathbf{u}_1, \mathbf{c})\frac{\delta H}{\delta \mathbf{u}_1} + \pi_{22}(\mathbf{u}_1, \mathbf{c})\frac{\delta H}{\delta \mathbf{u}_2} =: \pi^{\mathrm{red}}(\mathbf{u}_1)\frac{\delta H}{\delta \mathbf{u}_1}.$$

Hence the reduced tensor takes the form

$$\pi^{\mathrm{red}}(\mathbf{u}_1) = \pi_{11}(\mathbf{u}_1, \mathbf{c}) - \pi_{12}(\mathbf{u}_1, \mathbf{c}) \cdot [\pi_{22}(\mathbf{u}_1, \mathbf{c})]^{-1} \cdot \pi_{21}(\mathbf{u}_1, \mathbf{c}). \tag{60}$$

Lemma 2.15 [24]**.** *The operator (60) is a Poisson operator on the affine space $\mathcal{U}_1 \oplus \mathbf{c}$.*

The skew-symmetry of (60) is obvious. However, the proof of the Jacobi identity for (60) consists of tedious yet rather straightforward calculations [24] and we will omit it. In the finite-dimensional case, the said proof is much simpler (see for example [60]).

Note that if the inner product is not ad-invariant or we use the central extension procedure, then in general we do not know the explicit form of the Casimir functions (like (29) for example) for the natural Lie–Poisson bracket (or the extension thereof). In such a case one has to look for the annihilators $\mathrm{d}C$ of the Lie–Poisson tensor by directly solving the equation $\pi\,\mathrm{d}C = 0$. With $\mathrm{d}C$ in hand, one can try to reconstruct the Casimir functions C. The Poincaré lemma says that if the phase space $\mathcal{U}$ is linear or of star shape ($\forall\, u \in \mathcal{U}\,\{\varepsilon u : 0 \geqslant \epsilon \geqslant 1\} \subset \mathcal{U}$), then each closed k-form is exact. In particular, when $\mathcal{U}$ satisfies the condition of the Poincaré lemma, we can reconstruct the Casimir functions $C \in \mathcal{C}^\infty(\mathfrak{g}^*)$ from their differentials $\mathrm{d}C \in \mathfrak{g}$ using the homotopy formula [68]

$$C(\eta) = \int_0^1 \langle \mathrm{d}C(\epsilon\eta), \eta\rangle\, \mathrm{d}\epsilon \qquad \eta \in \mathfrak{g}^*. \tag{61}$$

Nevertheless, even when (61) is not applicable, the functions C can often be reconstructed through explicit integrations.

2.10. Lax hierarchies from loop algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ with the Lie bracket $[\cdot,\cdot]$. In contrast with section 2.2, we do not assume here any additional structures on $\mathfrak{g}$.

Definition 2.16. *The loop algebra over $\mathfrak{g}$ is the algebra $\mathfrak{g}^\lambda := \mathfrak{g}[[\lambda, \lambda^{-1}]]$ of formal Laurent series in the parameter $\lambda \in \mathbb{K}$ with the coefficients from $\mathfrak{g}$.*

It is easily seen that thanks to the bilinearity the operation in the former algebra extends to the loop algebra. Thus, in our case we can readily extend the Lie bracket $[\cdot,\cdot]$ to the loop algebra $\mathfrak{g}^\lambda$ by setting

$$[a\lambda^m, b\lambda^n] = [a, b]\lambda^{m+n} \qquad a, b \in \mathfrak{g} \quad m, n \in \mathbb{Z}. \tag{62}$$

There are two natural decompositions of $\mathfrak{g}^\lambda$ into the sum of Lie subalgebras, i.e.

$$\mathfrak{g}^\lambda = \mathfrak{g}^\lambda_+ \oplus \mathfrak{g}^\lambda_- = \left(\sum_{i\geqslant k} u_i\lambda^i\right) \oplus \left(\sum_{i<k} u_i\lambda^i\right) \tag{63}$$

for $k = 0$ and 1. Thus for $k = 0$ and $k = 1$ we have well-defined classical R-matrices (12)

$$R = P_+ - \tfrac{1}{2}. \tag{64}$$

The transformation $\lambda \mapsto \lambda^{-1}$ maps the case of $k = 0$ into that of $k = 1$, and vice versa. For this reason in what follows we restrict ourselves to considering the case of $k = 0$ only, and hence P_+ and P_- will stand for projections onto nonnegative and negative powers of λ. In fact, we have an infinite family of classical R-matrices

$$R_n = R\lambda^n \qquad n \in \mathbb{Z} \tag{65}$$

and the corresponding new Lie brackets on $\mathfrak{g}$ read

$$[a, b]_{R_n} := [R_n a, b] + [a, R_n b], \qquad a, b \in \mathfrak{g}. \tag{66}$$

The R-matrices (65) are well defined since λ^n is an intertwining operator and proposition 2.3 holds.

Let L be an element of $\mathfrak{g}$. We have the following Lax hierarchy:

$$L_{t_n} = [(\lambda^n L)_+, L] = -[(\lambda^n L)_-, L] \qquad n \in \mathbb{Z}. \tag{67}$$

The commutativity of the flows (67) for different n follows from the fact that (64) satisfies the Yang–Baxter equation (3) and $X_n(L) = \lambda^n L$ are invariants (5) such that $(X_m(L))_{t_n} = [RX_m(L), X_n(L)]$ still holds. Thus, the details of computations are parallel to those from the proof of proposition 2.4.

Without concentrating on the specific properties of $\mathfrak{g}$, we can investigate the general form of appropriate Lax operators from $\mathfrak{g}^\lambda$, i.e., the operators $L \in \mathfrak{g}^\lambda$ that generate self-consistent evolution equations on $\mathfrak{g}^\lambda$ from the Lax equations (67). This means that the maximal and minimal orders in λ of right- and left-hand sides of (67) have to coincide. Consider a bounded Lax operator $L \in \mathfrak{g}^\lambda$ of the form

$$L = u_N\lambda^N + u_{N-1}\lambda^{N-1} + \cdots + u_{1-m}\lambda^{1-m} + u_{-m}\lambda^{-m} \qquad N, m \in \mathbb{Z}, \tag{68}$$

where $N \geqslant -m$ and $u_i \in \mathfrak{g}$. Then a straightforward analysis shows that (68) yields consistent equations (67) if u_N is a nonzero time-independent element of $\mathfrak{g}$, i.e., $(u_N)_{t_n} = 0$. The specific properties of $\mathfrak{g}$ might lead to further restrictions on (68).

Theorem 2.7 can be applied for the construction of Hamiltonian hierarchies on the dual algebra to $\mathfrak{g}^\lambda$. We understand the dual algebra as $\mathfrak{g}^{*\lambda} = \mathfrak{g}^*[\![\lambda, \lambda^{-1}]\!]$. However, in contrast with the previously considered algebras, in the case of a loop algebra we have an infinite family of R-matrices (65) with respective Lie–Poisson brackets (21) and related Poisson tensors of the form $\pi_{R_n}\,\mathrm{d}H = \mathrm{ad}^*_{R_n\,\mathrm{d}H}\eta + R_n^*\mathrm{ad}^*_{\mathrm{d}H}\eta$, where $\eta \in \mathfrak{g}^{*\lambda}$ and $\mathrm{d}H \in \mathfrak{g}^\lambda$.

The coadjoint action ad^* is defined with respect to the Lie bracket (62) on $\mathfrak{g}^\lambda$. All the above Lie–Poisson brackets are mutually compatible, which follows from the fact that the sum of intertwining operators also is an intertwining operator. Besides, if $C_n \in \mathcal{C}^\infty(\mathfrak{g}^{*\lambda})$ is a Casimir function of the natural Lie–Poisson bracket (19), i.e. $\mathrm{ad}^*_{\mathrm{d}C_n}\eta = 0$, then $\lambda^l C_n$ is also a Casimir function. Hence, in the loop algebra case the Casimir functions generate multi-Hamiltonian hierarchies of mutually commuting vector fields on $\mathfrak{g}^{*\lambda}$

$$\eta_{t_n} = \mathrm{ad}^*_{R\mathrm{d}C_n}\eta = \cdots = \pi_{R_{-1}}\mathrm{d}C_{n+1} = \pi_R\,\mathrm{d}C_n = \pi_{R_1}\,\mathrm{d}C_{n-1} = \cdots,$$

where $\mathrm{d}C_{n+l} = \lambda^l\,\mathrm{d}C_n$ for $l \in \mathbb{Z}$.

If we have a trace form on $\mathfrak{g}$, given by a linear map $\mathrm{tr} : \mathfrak{g} \to \mathbb{K}$ such that the form in question is non-degenerate and symmetric, then this form can be extended to the loop algebra $\mathfrak{g}^\lambda$ by the formula

$$\mathrm{Tr}(a) = \mathrm{tr}(\mathrm{res}\, a) \qquad a \in \mathfrak{g}^\lambda, \tag{69}$$

where $\mathrm{res}\sum_i a_i\lambda^i = a_{-1}$. In fact, the choice of residue in (69) is a matter of convention, and one can choose the coefficient of an arbitrary order to get a proper definition of a trace. This is in contrast with the trace form (79) in the case of Poisson algebras. Nondegeneracy and symmetry of (69) are preserved. Moreover, if tr defines an ad-invariant metric on $\mathfrak{g}$, then this is also true for (69), and one can make an identification $\mathrm{ad}^* \equiv \mathrm{ad}$.

Consider now the central extension approach for the loop algebras. Assume for simplicity that on $\mathfrak{g}$, and therefore on $\mathfrak{g}^\lambda$, we have a nondegenerate inner product, and hence $\mathfrak{g}^{*\lambda} \cong \mathfrak{g}^\lambda$. Let $\widehat{\mathfrak{g}}^\lambda$ be an extension of $\mathfrak{g}^\lambda$ with the Lie bracket (41) defined by a 2-cocycle ω. The extended natural Lie–Poisson bracket has the form

$$\{H, F\}(L) := (L, [\mathrm{d}F, \mathrm{d}H])_{\mathfrak{g}^\lambda} + \alpha\omega(\mathrm{d}F, \mathrm{d}H). \tag{70}$$

In contrast with the previous case, we have here an infinite family of R-matrices (65) inducing an associated infinite family of new Lie–Poisson brackets on $\mathcal{C}^\infty(\mathfrak{g}^\lambda)$,

$$\{H, F\}_n(L) := (L, [\mathrm{d}F, \mathrm{d}H]_{R_n})_{\mathfrak{g}} + \alpha\omega_{R_n}(\mathrm{d}F, \mathrm{d}H), \tag{71}$$

where

$$\omega_{R_n}(a, b) := \omega(R_n a, b) + \omega(a, R_n b) \qquad a, b \in \mathfrak{g}^\lambda. \tag{72}$$

All quantities (72) are 2-cocycles of the respective Lie brackets (66). For $n = 0$ according to proposition 2.13 this follows from the fact that R satisfies the classical Yang–Baxter equation, in the remaining cases one must additionally use the fact that λ^n are intertwining operators.

It is important to stress that all Poisson brackets (71) are pairwise compatible, which follows from the fact that the linear sum of intertwining operators is an intertwining operator. If the 2-cocycle ω is given in the form (44), then the Casimirs of the extended natural Lie–Poisson bracket (70) satisfy

$$\pi\,\mathrm{d}C_n = \mathrm{ad}^*_{\mathrm{d}C_n}L + \alpha\phi(\mathrm{d}C_n) = 0, \tag{73}$$

and generate multi-Hamiltonian Lax hierarchy

$$L_{t_n} = \mathrm{ad}^*_{R\,\mathrm{d}C_n}L + \alpha\phi(R\,\mathrm{d}C_n) = \cdots = \pi_{R_{-1}}\,\mathrm{d}C_{n+1} = \pi_R\,\mathrm{d}C_n = \pi_{R_1}\,\mathrm{d}C_{n-1} = \cdots, \tag{74}$$

where $L \in \mathfrak{g}^\lambda$, with the restriction that $\mathrm{d}C_{n+l} = \lambda^n\,\mathrm{d}C_n$ for $l \in \mathbb{Z}$. The respective Poisson tensors associated with the brackets (71) are given by

$$\pi_{R_l}\,\mathrm{d}H = \mathrm{ad}^*_{R_l\,\mathrm{d}H}L + R_l^*\mathrm{ad}^*_{\mathrm{d}H}L + \alpha\phi(R_l\,\mathrm{d}H) + \alpha R_l^*\phi(\mathrm{d}H). \tag{75}$$

We readily see that $R_l^\star = -\lambda^l R$. To find $\mathrm{d}C_n$, we can assume that

$$\mathrm{d}C_n \equiv \lambda^n\,\mathrm{d}C_0 = a_0 + a_1\lambda^1 + a_2\lambda^{-2} + \cdots \qquad n \in \mathbb{Z},$$

and solve (73) recursively for the coefficients a_i. Then the functions C_n in principle can be reconstructed using the homotopy formula (61).

Consider now a particular case of the Maurer–Cartan 2-cocycle (44), with $\phi = \partial_x$ and the loop algebra $\widetilde{\mathfrak{g}}^\lambda$ over (54). Assuming that $\mathrm{ad}^* \equiv \mathrm{ad}$, the Lax hierarchy (74) takes the form

$$L_{t_n} = [(\mathrm{d}C_n)_+, L] + \alpha\partial_x(\mathrm{d}C_n)_+ = \cdots = \pi_{R_l}\,\mathrm{d}C_{n-l} = \cdots. \tag{76}$$

Then, analyzing (76) we find that $L \in \widetilde{\mathfrak{g}}^\lambda$ of the form (68) yields self-consistent equations for $\alpha \neq 0$ if $N \geqslant -1$ and u_N is nonzero and time-independent (except for $N = -1$) and $m \geqslant 0$. The Hamiltonian structures for (76) are given by the Poisson tensors (75) taking the form

$$\pi_{R_l}\,\mathrm{d}H = [(\lambda^l\,\mathrm{d}H)_+, L] - \lambda^l[\mathrm{d}H, L]_+ + \alpha\partial_x(\lambda^l\,\mathrm{d}H)_+ - \alpha\lambda^l\partial_x(\mathrm{d}H)_+. \tag{77}$$

The Poisson tensors (77) form a proper subspace of $\widetilde{\mathfrak{g}}^\lambda$ with respect to (68), with the above restrictions, if $N \geqslant l \geqslant -m$ for $N \geqslant 0$ and if $0 \geqslant l \geqslant -m$ for $N = -1$. Thus, there always exist at least two Poisson tensors π_{R_l} for which the procedure of Dirac reduction is not required. Recall that this analysis disregards specific properties of the Lie algebra $\mathfrak{g}$. Thus, if L is further constrained according to these specific properties of $\mathfrak{g}$ the Dirac reduction might be yet required.

Finally, let us note that in contrast with the previously considered algebras, where the central extension has lead to (2+1)-dimensional systems, in the case of loop algebras the central extension is necessary for the construction of (1+1)-dimensional integrable continuous field systems. The reader will find the examples of this construction in the following sections.

A natural choice is loop algebras defined over finite-dimensional semi-simple Lie algebras. In such a case the Killing form gives us symmetric, nondegenerate and also ad-invariant inner product. Thus, taking into consideration the central extension procedure with the Maurer–Cartan 2-cocycle, the above choice leads to the constructions of a wide class of (1+1)-dimensional integrable continuous systems. The simplest case is presented in section 4.3.

3. Integrable dispersionless systems

The theory of integrable dispersionless or equivalently integrable hydrodynamic-type systems, i.e., the quasi-linear systems of first-order partial differential equations, belongs to the most recent ones and has been systematically developed from the 1980s. Significant progress was achieved after Tsarev [100] discovered a technique called the generalized hodograph method that permits us to find solutions using quadratures, see also [34].

The study of the Poisson structures of dispersionless systems was initiated by Dubrovin and Novikov [29]. They established a remarkable result that the Poisson tensors of hydrodynamic type can be generated by contravariant nondegenerate flat Riemannian metrics. The natural geometric setting for the associated bi-Hamiltonian structures (Poisson pencils) is the theory of Frobenius manifolds based on the geometry of pencils of contravariant Riemannian metrics [28]. The Frobenius manifolds were introduced by Dubrovin as a coordinate-free form of the associativity equations, appearing in the context of deformations of two-dimensional topological field theories (TFT), the so-called WDVV equations, that can be identified with (a class of) hydrodynamic-type systems.

3.1. Poisson algebras of Laurent series

Consider the algebra of 'formal' Laurent series in $p \in \mathbb{C}^*$ about ∞,

$$\mathcal{A} = \mathcal{A}_{\geqslant k-r} \oplus \mathcal{A}_{<k-r} := \left\{ \sum_{i=k-r}^{N} a_i(x) p^i \right\} \oplus \left\{ \sum_{i<k-r} a_i(x) p^i \right\},$$

where u_i are smooth functions of continuous variable $x \in \Omega$, i.e., $\mathcal{A}$ consists of polynomial functions in p and p^{-1} with finite highest orders, where $\Omega = \mathbb{S}^1$ if we assume these functions to be periodic in x or $\Omega = \mathbb{R}$ if these functions belong to the Schwartz space (u_i and all their derivatives tend rapidly to zero when x approaches $\pm\infty$). We can introduce the Lie algebra structure on $\mathcal{A}$ in infinitely many ways using a family of Poisson brackets

$$\{f, g\}_r := p^r \left(\frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} \right) \qquad r \in \mathbb{Z} \quad f, g \in \mathcal{A} \tag{78}$$

that generalize the well-known canonical Poisson bracket (the case $r = 0$).

The trace form in the algebra $\mathcal{A}$ with fixed Poisson bracket (78) for some r is defined in the following fashion:

$$\operatorname{Tr} f := -\int_\Omega \operatorname{res}_\infty (p^{-r} f)\, \mathrm{d}x \qquad f \in \mathcal{A}. \tag{79}$$

Here res is the standard residue at $p = \infty$ such that $\operatorname{res}_\infty L = -u_{-1}$ for $L = \sum_i u_i p^i$.

Proposition 3.1. *The scalar product (27) defined by means of the trace form (79) is symmetric, nondegenerate and ad-invariant.*

Proof. The nondegeneracy and symmetry are obvious. Let γ be a closed curve encircling once an infinity point on the extended complex plane. Then

$$\begin{aligned} \operatorname{Tr}\{f, g\}_r &= -\int_\Omega \operatorname{res}_\infty (\partial_p f \partial_x g)\, \mathrm{d}x + \int_\Omega \operatorname{res}_\infty (\partial_x f \partial_p g)\, \mathrm{d}x \\ &= -\frac{1}{2\pi \mathrm{i}} \int_\Omega \oint_{\gamma_\lambda} (\partial_p \partial_x g)\, \mathrm{d}p\, \mathrm{d}x + \frac{1}{2\pi \mathrm{i}} \int_\Omega \oint_{\gamma_\lambda} (\partial_x f \partial_p g)\, \mathrm{d}p\, \mathrm{d}x = 0, \end{aligned}$$

where the latter equality follows from integrations by parts with respect to p and x. Hence, the ad-invariance is a consequence of lemma (2.8). □

Obviously, (78) is a derivation for the multiplication in $\mathcal{A}$ and hence we can further apply the scheme from section 2.5. Fixing r we fix Poisson algebra and we are able to construct R-matrices following from the decomposition of $\mathcal{A}$. Simple inspection shows that $\mathcal{A}_{\geqslant k-r}$ and $\mathcal{A}_{<k-r}$ are Lie subalgebras of $\mathcal{A}$ only in the following cases:

- if $r = 0$ for $k = 0$;
- if $r \in \mathbb{Z}$ for $k = 1, 2$;
- if $r = 2$ for $k = 3$.

Thus, fixing r we fix the Lie algebra structure with k numbering the R-matrices given in the form $R = P_{\geqslant k-r} - \frac{1}{2}$. Hence, we have multi-Hamiltonian Lax hierarchies (33) of the form

$$L_{t_n} = \left\{(L^{\frac{n}{N}})_{\geqslant k-r}, L\right\}_r = \pi_0 \,\mathrm{d}H_n = \pi_1 \,\mathrm{d}H_{n-1} \qquad n = 1, 2, \ldots, \tag{80}$$

generated by fractional powers of infinite-field Lax functions $L \in \mathcal{A}$ given in the form

$$L = u_N p^N + u_{N-1} p^{N-1} + u_{N-2} p^{N-2} + \cdots \qquad N \neq 0, \tag{81}$$

where $u_N = 1, u_{N-1} = 0$ for $k = 0$ and $u_N = 1$ for $k = 1$. A simple analysis of (80) shows that (81) are appropriate Lax functions except for the case $k = 3$, which is excluded from further considerations. This means that all the Lax functions under study have pole or root at infinity.

We find that $R^* = \frac{1}{2} - P_{\geqslant 2r-k}$. The differentials of a given functional $H \in \mathcal{C}^\infty(\mathcal{A})$ of (81) have the form $\mathrm{d}H = \sum_{i=-\infty}^{N+k-2} \frac{\delta H}{\delta u_i} p^{r-1-i}$, such that (16) holds. Hence, the related Poisson tensors for (80) are given by (32)

$$\pi_q \,\mathrm{d}H = \{(L^q \,\mathrm{d}H)_{\geqslant k-r}, L\}_r - L^q(\{\mathrm{d}H, L\}_r)_{\geqslant 2r-k}, \tag{82}$$

where $q = 0, 1, \ldots,$ with the following Hamiltonians (29)

$$H_n = -\frac{N}{n+N} \operatorname{Tr}\left(L^{\frac{n}{N}+1}\right) \qquad n \neq -N. \tag{83}$$

We still have to check whether the above Lax functions span proper subspaces, with respect to the above Poisson operators (82), of the full Poisson algebras. We will restrict our considerations to linear ($n = 0$) and quadratic ($n = 1$) Poisson tensors, as they are obvious enough to define bi-Hamiltonian structures. Besides, in all nontrivial cases, the Lax functions do not span proper subspaces w.r.t. the Poisson tensors for $n \geqslant 2$.

For $k = 0$ the above Lax functions always span the proper subspace w.r.t. the linear Poisson tensor, but for $k = 1, 2$ this is the case only if $N \geqslant 2r - 2k + 1$, otherwise the Dirac reduction is required. For the quadratic Poisson tensors the Dirac reduction is always necessary. The reduced quadratic Poisson tensor for $k = r = 0, 1, 2$ is given by [95]

$$\pi_1^{\mathrm{red}} \,\mathrm{d}H = \{(L \,\mathrm{d}H)_{\geqslant 0}, L\}_r - L(\{\mathrm{d}H, L\}_r)_{\geqslant r} + \frac{1}{N}\left\{L, \partial_x^{-1} \mathrm{res}_\infty \{\mathrm{d}H, L\}_0\right\}_r, \tag{84}$$

see section 2.9 and lemma (2.15). The reduced Poisson tensors (84) are always local as the residue from the last term is always a total derivative.

The dispersionless systems with the Lax representations of the form (80) where $r = 0$ with $k = 0, 1$ (the canonical Poisson bracket) related to the dispersionless KP hierarchy and the dispersionless modified KP hierarchy and the case $k = r = 1$ of the dispersionless Toda hierarchy, together with their finite-field reductions, were considered in many papers (see for example [5, 17, 32, 48, 51, 54, 89, 99, 102] and more recently [19, 20, 71]). The theory with the Poisson bracket (78) for arbitrary integer r, from the point of view of classical R-matrices, was considered for the first time in [9] and further developed in [13, 95].

The decomposition of the algebra $\mathcal{A}$ into Lie subalgebras is preserved under the transformation $p \mapsto p^{-1}$ (the case k, r goes to $k' = 3 - k, r' = 2 - r$), but the Laurent series at ∞ (81) transform into Laurent series at zero. This fact suggests a more analytic approach to the construction of dispersionless systems, see [42, 52]. For the theory of meromorphic Lax representations (80) of dispersionless systems see [95]. In [52] Krichever introduced the so-called universal Whitham hierarchies by means of the moduli spaces of Riemann surfaces of all genera. Other classes of reductions yielding nonstandard integrable dispersionless systems can be found for instance in [41, 73, 101]. For a more detailed discussion of the Lax representations (80) for noncanonical Poisson brackets, their reductions and Hamiltonian structures as well as several examples of finite-dimensional reductions see [13, 95].

We know that the quantization of Poisson algebras for $r = 0$ (and for the equivalent case $r = 2$) gives the algebra of pseudo-differential operators and leads to the construction of field soliton systems, while the quantization of the case for $r = 1$ gives the algebra of shift operators and leads to the lattice soliton systems. However, the class of reductions yielding construction of dispersionless systems is much wider then the class of corresponding reductions yielding systems with dispersion. Besides, the issue of quantization of Poisson algebras for $r \neq 0, 1, 2$ that would lead to the construction of dispersive integrable systems is still open, see [15, 21, 90, 96] and references therein.

Below we present two examples of hydrodynamic chains (infinite-field systems) with their bi-Hamiltonian structures. For the classifications of hydrodynamic chains and related hydrodynamic Poisson tensors see [70–72].

Example 3.2. Let us consider the infinite-field Lax function (81) for $N = 1$: $L = p + \sum_{i\geqslant 0} u_i p^{-i-1}$. Then we find the first nontrivial hydrodynamic chain from hierarchy (80), the well-known Benney moment chain,

$$L_{t_2} = \tfrac{1}{2}\{(L^2)_{\geqslant 0}, L\}_0 = \pi_0\,\mathrm{d}H_2 = \pi_1\,\mathrm{d}H_1 \iff (u_i)_{t_2} = (u_{i+1})_x + \mathrm{i}u_{i-1}(u_0)_x,$$

where $(L^2)_{\geqslant 0} = p^2 + 2u_0$. The bi-Hamiltonian structure is given by the Poisson tensors

$$\begin{aligned}
\pi_0^{ij} &= j\partial_x u_{i+j-1} + iu_{i+j-1}\partial_x \\
\pi_1^{ij} &= (i+1)u_{i+j}\partial_x + (j+1)\partial_x u_{i+j} + (i+1)ju_{i-1}\partial_x u_{j-1} \\
&\quad + \sum_{k=0}^{j-1}[(i-j+k)u_{i-1+k}\partial_x u_{j-1-k} + ku_{j-1-k}\partial_x u_{i-1+k}],
\end{aligned}$$

which are obtained from (82) for $q = 0$ and (84). The respective Hamiltonians are (83)

$$H_1 = \frac{1}{2}\int_\Omega u_1\,\mathrm{d}x \qquad H_2 = \frac{1}{2}\int_\Omega \left(u_0^2 + u_2\right)\mathrm{d}x.$$

Example 3.3. The case of $k = r = 1$ with the Lax function (81) for $N = 1$ of the form $L = p + \sum_{i=0}^{\infty} u_i p^{-i}$. The first hydrodynamic chain from (80) has the following form:

$$L_{t_1} = \{(L)_{\geqslant 0}, L\}_1 = \pi_0\,\mathrm{d}H_1 = \pi_1\,\mathrm{d}H_0 \iff (u_i)_{t_1} = (u_{i+1})_x + iu_i(u_0)_x.$$

The bi-Hamiltonian structure is given by the Poisson tensors

$$\begin{aligned}
\pi_0^{ij} &= j\partial_x u_{i+j} + iu_{i+j}\partial_x \\
\pi_1^{ij} &= \sum_{k=0}^{i}[(j-k)u_k\partial_x u_{i+j-k} + (i-k)u_{i+j-k}\partial_x u_k] + i(j+1)u_i\partial_x u_j \\
&\quad + (j+1)\partial_x u_{i+j+1} + (i+1)u_{i+j+1}\partial_x
\end{aligned}$$

as well as the Hamiltonians $H_0 = \int_\Omega u_0\,\mathrm{d}x$ and $H_1 = \int_\Omega \left(u_1 + \frac{1}{2}u_0^2\right)\mathrm{d}x$.

The central extension approach from section 2.8 with the Mauren–Cartan 2-cocycle yields the following Lax hierarchy (58) of (2+1)-dimensional hydrodynamic systems

$$L_{t_n} = \{(\mathrm{d}C_n)_{\geqslant k-r}, L - \alpha q\}_r = \pi_0\,\mathrm{d}H_n \qquad n = 1, 2, \ldots, \tag{85}$$

where the Poisson bracket takes the form

$$\{f, g\}_r := p^r \left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial x}\right) + \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial q}\right) \qquad r \in \mathbb{Z} \quad f, g \in \widetilde{\mathcal{A}}.$$

The infinite-field Lax operators (81), with u_i depending additionally on $y \in \mathbb{S}^1$, are admissible with respect to (85) if $N \geqslant 1 - r$. To construct evolution equations from (85), we take $\mathrm{d}C_n = \sum_{i=0}^{\infty} a_{n-i} p^{n-i}$ and solve the Novikov–Lax equation (57), $\{\mathrm{d}C_n, L - \alpha q\}_r = 0$ for the auxiliary fields a_i in terms of the fields from Lax function. The linear Poisson tensor (51) is given by

$$\pi_0\,\mathrm{d}H = \{(\mathrm{d}H)_{\geqslant k-r}, L - \alpha q\}_r - (\{\mathrm{d}H, L - \alpha q\}_r)_{\geqslant 2r-k}.$$

The Lax functions (81) span the proper subspace w.r.t. the above linear Poisson tensor if $N \geqslant 2r - 2k + 1$, otherwise the Dirac reduction is required. The construction of (2+1)-dimensional dispersionless systems by means of central extension procedure yielding the Lax hierarchy was presented in [14], where one can also find a number of examples.

Example 3.4. The case of $k = r = 0$. Consider $L = p^2 + u$. Then, for $(\mathrm{d}C_3)_{\geqslant 0} = p^3 + \frac{3}{2}up + \frac{3}{4}\alpha\partial_x^{-1}u_y$ we obtain the (2+1)-dimensional dKP equation

$$L_{t_3} = \{(\mathrm{d}C_3)_{\geqslant 0}, L - \alpha q\}_0 = \pi_0\,\mathrm{d}H_3 = \pi_1\,\mathrm{d}H_1 \iff u_{t_3} = \tfrac{3}{2}uu_x + \tfrac{3}{4}\alpha^2\partial_x^{-1}u_y,$$

where the Poisson tensors are $\pi_0 = 2\partial_x$ and if $\alpha = 0 : \pi_1 = \partial_x u + u\partial_x$. The Hamiltonians are

$$H_1 = \iint_{\Omega\times\mathbb{S}^1} \frac{1}{4}u^2\,\mathrm{d}x\,\mathrm{d}y \qquad H_3 = \iint_{\Omega\times\mathbb{S}^1} \frac{1}{16}\left(2u^3 + 3\alpha^2 u\partial_x^{-2}u_{yy}\right)\mathrm{d}x\,\mathrm{d}y.$$

Example 3.5. Consider the Lax operator $L = p^{2-r} + up^{1-r} + vp^{-r}$ for $k = 1$ and $r \in \mathbb{Z}, r \neq 2$. Then for $(\mathrm{d}C_{2-r})_{\geqslant -r+1} = p^{2-r} + up^{1-r}$ we have

$$L_{t_{2-r}} = \{(\mathrm{d}C_{2-r})_{\geqslant 1-r}, L - \alpha q\}_r = \pi_0\,\mathrm{d}H_{2-r} = \pi_1\,\mathrm{d}H_{1-r}$$

$$\iff \begin{pmatrix} u \\ v \end{pmatrix}_{t_{2-r}} = \begin{pmatrix} (2-r)v_x + \alpha u_y \\ ru_x v + (1-r)uv_x \end{pmatrix}.$$

For $r = 1$ we get the (2+1)-dimensional dispersionless Toda equation with Hamiltonian structure given by

$$\pi_0 = \begin{pmatrix} \alpha\partial_y & \partial_x v \\ v\partial_x & 0 \end{pmatrix} \qquad \text{and} \qquad \text{if} \quad \alpha = 0, \quad \pi_1 = \begin{pmatrix} \partial_x v + v\partial_x & u\partial_x v \\ v\partial_x u & 2v\partial_x v \end{pmatrix},$$

with the Hamiltonians

$$H_1 = \iint_{\Omega\times\mathbb{S}^1} \left(v + \frac{1}{2}u^2\right)\mathrm{d}x\,\mathrm{d}y \qquad H_0 = \iint_{\Omega\times\mathbb{S}^1} u\,\mathrm{d}x\,\mathrm{d}y.$$

3.2. Universal hierarchy

Let $\mathfrak{g} = \mathrm{Vect}(\mathbb{S}^1)$ be the Lie algebra of (smooth) vector fields on the circle $\mathbb{S}^1$ over the field $\mathbb{K}$. The elements of $\mathrm{Vect}(\mathbb{S}^1)$ can be identified with smooth functions $a(x)$ of spatial variable $x \in \mathbb{S}^1$, with a Lie bracket in $\mathfrak{g}$ of the form

$$\langle a, b\rangle := ab_x - ba_x, \tag{86}$$

where $a, b \in \mathfrak{g}$. Note that (86) is a well-defined Lie bracket that does not satisfy the Leibniz rule (4). Thus, we follow the scheme from section 2.10 and consider the loop algebra over $\mathrm{Vect}(\mathbb{S}^1)$, i.e., $\mathfrak{g}^\lambda = \mathrm{Vect}(\mathbb{S}^1)[\![\lambda, \lambda^{-1}]\!]$. The commutator (86) readily extends to $\mathfrak{g}^\lambda$. We already know that we have the decomposition of $\mathfrak{g}^\lambda$ into Lie subalgebras (63), with respective classical R-matrices generating the following Lax hierarchies (67):

$$L_{t_n} = \langle(\lambda^n L)_{\geqslant k}, L\rangle \qquad k = 0, 1, \tag{87}$$

where $L \in \mathfrak{g}^\lambda$ and $n \in \mathbb{Z}$.

The hierarchy (87) is the so-called universal hierarchy of hydrodynamic type, which has been a subject of intensive research in recent years [61, 62]. The hierarchy (87) can be obtained as a quasi-classical limit of the coupled KdV equations of Antonowicz and Fordy [4]. In this fashion we can obtain multi-Hamiltonian structure for the universal hierarchy [36]. In fact, the multi-Hamiltonian structure of the coupled KdV equations using classical R-matrix approach can be algebraically interpreted as a set of compatible Lie–Poisson structures on the dual space to the loop Virasoro algebra [38]. The Virasoro algebra is a central extension of the Lie algebra of vector fields on a circle $\mathrm{Vect}(\mathbb{S}^1)$ associated with the Gelfand–Fuchs 2-cocycle (i.e. (44) with $\phi = \partial_x^3$).

Example 3.6. Consider (87) for the infinite-field Lax functions given in the following (appropriate) form $L = u_0 + u_1\lambda^{-1} + u_2\lambda^{-2} + \cdots$, where $u_0 = 1$ for $k = 0$. One can observe that $L_{t_n} = \langle(\lambda^n L)_{\geqslant k}, (\lambda^n L)_{<k}\rangle\lambda^{-n}$. Hence, the evolution equations from (87) take the form of (1+1)-dimensional hydrodynamic chains

$$(u_i)_{t_n} = (1-k)(u_i)_x + \sum_{j=1-k}^{i-1+k} \langle u_{i-j}, u_{n+j}\rangle \qquad k = 0, 1.$$

Assume that the dynamical fields in $\mathfrak{g}^\lambda$ depends on an additional spatial variable y. Let us now consider a (2+1)-dimensional counterpart of (87). It reads

$$L_{t_n} = \langle(A_n)_{\geqslant k}, L\rangle + \partial_y (A_n)_{\geqslant k} \qquad k = 0, 1, \tag{88}$$

where $A_n = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots$ satisfy

$$\langle A_n, L\rangle + (A_n)_y = 0. \tag{89}$$

Note that $A_n = \lambda^n A_0$. For a given $L \in \mathfrak{g}^\lambda$ one finds coefficients a_i from A_n by solving (89) recursively. Note that one cannot obtain (88) as a central extension of the universal hierarchy, as (56) is not a 2-cocycle associated with $\mathrm{Vect}(\mathbb{S}^1)$. Commutativity of the equations from the hierarchy (88) can be proved by straightforward computation. On the other hand, integrability of the equations from (88) follows from the fact that (88) is a Lax sub-hierarchy of the centrally extended cotangent universal hierarchy considered in [85].

Example 3.7. Let L have the form $L = \lambda + u$ in the case of $k = 0$. Then solving (89) one finds that $A_0 = 1 + u\lambda^{-1} + \partial_x^{-1}u_y\lambda^{-2} + \cdots$. Hence, the first nontrivial (2+1)-dimensional hydrodynamic equation from the hierarchy (88) is [62]

$$L_{t_2} = \langle(A_2)_{\geqslant 0}, L\rangle + \partial_y (A_n)_{\geqslant 0} \quad\Longleftrightarrow\quad u_{t_2} = \partial_x^{-1}u_{yy} - uu_y + u_x\partial_x^{-1}u_y.$$

This hydrodynamic equation is equivalent to a system of the form

$$u_{t_2} - v_y + uv_x - u_x v = 0 \qquad v_x - u_y = 0,$$

that has recently attracted considerable attention, see [30, 35, 59, 69, 70, 85].

Example 3.8. The case of $k = 1$ for $L = u\lambda^{-1}$. We have $A_0 = 1 - \partial_y^{-1}u_x\lambda^{-1} + \cdots$. Hence

$$L_{t_2} = \langle(A_2)_{\geqslant 1}, L\rangle + \partial_y (A_n)_{\geqslant 1} \quad\Longleftrightarrow\quad u_{t_2} = u\partial_y^{-1}u_{xx} - u_x\partial_y^{-1}u_x.$$

4. Integrable dispersive systems

In the following section, we apply the general formalism to several infinite-dimensional Lie algebras in order to construct a vast family of integrable dispersive (continuous and discrete soliton) systems.

Recently, the so-called integrable q-analogues of KP and Toda-like hierarchies have become of increasing interest, see [3, 39, 45, 46, 98]. Our approach presented in two following subsections includes the q-systems as a special case. Actually, we consider generalized algebras of shift and pseudo-differential operators that allow us to construct in one scheme not only ordinary lattice and field systems, but in particular also their q-deformations.

We also present (2+1)-dimensional extensions of the Lax hierarchies following from the algebras of shift and pseudo-differential operators. In these cases, the quadratic Poisson tensor is not preserved. Nevertheless, the second Poisson tensor can be given by means of the so-called operand formalism [37, 57, 80]. Its construction, within classical R-matrix approach, can be found in [16].

In the last subsection, we present the application of classical R-matrix formalism to the loop algebra $sl(2,\mathbb{C})[[\lambda,\lambda^{-1}]]$, which is the simplest case of infinite-dimensional Lie algebras of Kac–Moody. In fact the scheme of the construction of infinite-dimensional systems from affine Kac-Moody Lie algebras is one of the most general and particular cases are closely related to the algebras of shift and pseudo-differential operators yielding soliton systems (see for instance [22, 33]). Note that since the affine Kac–Moody Lie algebras are central extensions of loop algebras defined over finite-dimensional semi-simple Lie algebras, the presented formalism in section 2.10 is sufficient for the application in this case. For details we send the reader to the important review [27].

For the application of R-matrix formalism to other algebras see for example [75–77, 87]. A specially interesting class of algebras is related to the so-called super-symmetric (SUSY) systems. The reader will find the details in [18, 44, 65, 74, 76] and in literature quoted there.

4.1. Algebra of shift operators

Consider the algebra of 'formal' shift operators

$$\mathfrak{g} = \mathfrak{g}_{\geqslant k-1} \oplus \mathfrak{g}_{<k-1} = \left\{ \sum_{i\geqslant k-1}^{N} u_i\, \mathcal{E}^i \right\} \oplus \left\{ \sum_{i<k-1} u_i\, \mathcal{E}^i \right\}, \tag{90}$$

where $u_i \in \mathcal{F}$, and $\mathcal{F}$ is an algebra of dynamical fields with values in $\mathbb{K}$. The associative multiplication rule in $\mathfrak{g}$ (90) is defined by

$$\mathcal{E}^m u = E^m(u)\mathcal{E}^m \qquad m \in \mathbb{Z}. \tag{91}$$

Proposition 4.1. *The multiplication in (90) of the form (91) is associative if and only if $E : \mathcal{F} \to \mathcal{F}$ is an invertible endomorphism (automorphism), i.e.*

$$E(uv) = E(u)E(v).$$

The proof is straightforward. The Lie bracket in $\mathfrak{g}$ is given by the commutator $[A, B] = AB - BA$, where $A, B \in \mathfrak{g}$.

Let $\operatorname{tr} : \mathfrak{g} \to \mathbb{K}$ be a trace form, being a linear map, such that

$$\operatorname{tr}(A) := \langle \operatorname{free}(A)\rangle, \tag{92}$$

where $\mathrm{free}(A) := a_0$ for $A = \sum_i a_i \mathcal{E}^i$ and $\langle\cdot\rangle$ denotes a functional whose form depends on the realization of the algebra $\mathcal{F}$ and the endomorphism E. We assume that $\langle\cdot\rangle$ is such that

$$\langle Ef\rangle = \langle f\rangle \qquad f \in \mathcal{F} \tag{93}$$

holds. Note that from this assumption it follows that $\langle uEv\rangle = \langle vE^{-1}u\rangle$, thus the adjoint of operator E is $E^\dagger = E^{-1}$.

Proposition 4.2. *The bilinear map defined as*

$$(A, B)_{\mathfrak{g}} := \mathrm{tr}(AB) \tag{94}$$

is an inner product on $\mathfrak{g}$ which is nondegenerate, symmetric and ad-invariant.

Proof. The nondegeneracy of (94) is obvious. The symmetricity follows from the definition by using (93). The ad-invariance is a consequence of the associativity of multiplication operation in $\mathfrak{g}$, see lemma 2.8. □

Let us mention now a few standard realizations of the algebra of shift operators (90):

- The first one is given by the shift operators on a *discrete lattice*. In this case, the dynamical functions are $u_i : \mathbb{Z} \to \mathbb{K}$ and $E^m u(n) := u(n+m)$. The form of the functional from (92) is $\langle f\rangle := \sum_{n\in\mathbb{Z}} f(n)$.
- The second realization is given by the shift operators on a *continuous lattice*. Now $u_i : \Omega \to \mathbb{K}$ are smooth functions of x, where $\Omega = \mathbb{S}^1$ or $\Omega = \mathbb{R}$, if u_i are from the Schwartz space. In this case the shift operator is $E^m u(x) := u(x + m\hbar)$, where $\hbar$ is a parameter. In this case the functional is just integration, i.e., $\langle f\rangle := \int_\Omega f(x)\,\mathrm{d}x$.
- The third realization is for *q-discrete* functions $u_i : \mathbb{K}_q \to \mathbb{K}$, where $\mathbb{K}_q := q^{\mathbb{Z}} \cup \{0\}$ for $q \neq 0$. Here $E^m u(x) := u(q^m x)$ and $\langle f\rangle := \sum_{n\in\mathbb{Z}} f(q^n)$.

All the above realizations lead to the construction of lattice soliton systems: discrete, continuous and q-discrete, respectively. More realizations can be made by means of the discrete one-parameter groups of diffeomorphisms [10] or by means of jump operators on timescales [12, 43]. Note that in the continuous limit the algebra of shift operators on lattices gives the Poisson algebra for (78) with $r = 1$. Thus, the quasi-classical limit of lattice soliton systems are respective dispersionless systems. The situation in the q-discrete case is similar [10].

The subspaces $\mathfrak{g}_{\geqslant k-1}$ and $\mathfrak{g}_{<k-1}$ of (90) are Lie subalgebras only for $k = 1$ and $k = 2$ and the classical R-matrices following from the decomposition of $\mathfrak{g}$ are (12) $R = P_{\geqslant k-1} - \frac{1}{2}$. Their adjoints with respect to the above inner product are given by $R^* = P_{<2-k} - \frac{1}{2}$, respectively.

As a result, we have two Lax hierarchies:

$$L_{t_n} = \left[\left(L^{\frac{n}{N}}\right)_{\geqslant k-1}, L\right] = \pi_0\,\mathrm{d}H_n = \pi_1\,\mathrm{d}H_{n-1} \qquad k = 1, 2, \tag{95}$$

of infinitely many mutually commuting systems. Let (95) be generated by powers of appropriate Lax operators $L \in \mathfrak{g}$ of the form

$$L = u_N\,\mathcal{E}^N + u_{N-1}\,\mathcal{E}^{N-1} + u_{N-2}\,\mathcal{E}^{N-2} + u_{N-3}\,\mathcal{E}^{N-3} + \cdots, \tag{96}$$

where $u_N = 1$ for $k = 1$.

The bi-Hamiltonian structure of the Lax hierarchies (95) is defined by the compatible (for fixed k) Poisson tensors given by formulae (34),

$$\pi_0\,\mathrm{d}H = [L, (\mathrm{d}H)_{<k-1}] + ([\mathrm{d}H, L])_{<2-k}$$

and

$$\begin{aligned}\pi_1\,\mathrm{d}H = {}&\tfrac{1}{2}([L, (L\,\mathrm{d}H + \mathrm{d}H L)_{<k-1}] + L([\mathrm{d}H, L])_{<2-k} + ([\mathrm{d}H, L])_{<2-k} L)\\ &+ (2-k)[(E+1)(E-1)^{-1}\,\mathrm{free}([\mathrm{d}H, L]), L],\end{aligned} \tag{97}$$

where the operation $(E-1)^{-1}$ is the formal inverse of $(E-1)$. The second Poisson tensor is a Dirac reduction of (38) as in this case $\widetilde{R} = R + \left(k - \frac{3}{2}P_0\right)$ satisfies the related condition.

The differentials $\mathrm{d}H(L)$ of functionals $H(L) \in \mathcal{C}^\infty(\mathfrak{g})$ for (96) have the form $\mathrm{d}H = \sum_{i=2-k}^{\infty} \mathcal{E}^{i-N} \frac{\delta H}{\delta u_{N-i}}$ and the respective Hamiltonians (29) are

$$H_n(L) = \frac{N}{N+n}\,\mathrm{tr}\left(L^{\frac{n}{N}+1}\right) \qquad N \neq -n.$$

The theory of lattice soliton systems of Toda type with Lax representations given by means of the shift operators was introduced for the first time by Kupershmidt in [53]. This class of systems was investigated, from the point of view of the classical R-matrix formalism applied to the algebra of shift operators, in [11, 64].

Example 4.3. Consider the case of $k = 1$ with (96) normalized as $L = \mathcal{E} + \sum_{i=0}^{\infty} u_i \mathcal{E}^{-i}$. The first chain from the Lax hierarchy (95) has the form

$$L_{t_1} = [(L)_{\geqslant 0}, L] = \pi_0\,\mathrm{d}H_1 = \pi_1\,\mathrm{d}H_0 \iff (u_i)_{t_1} = (E-1)u_{i+1} + u_i(1 - E^{-i})u_0, \tag{98}$$

and its explicit bi-Hamiltonian structure is given by

$$\begin{aligned}
\pi_0^{ij} &= E^j u_{i+j} - u_{i+j} E^{-i} \\
\pi_1^{ij} &= \sum_{k=0}^{i} [u_k E^{j-k} u_{i+j-k} - u_{i+j-k} E^{k-i} u_k + u_i (E^{j-k} - E^{-k}) u_j] \\
&\qquad + u_i (1 - E^{j-i}) u_j + E^{j+1} u_{i+j+1} - u_{i+j+1} E^{-i-1}
\end{aligned}$$

together with the Hamiltonians $H_0 = \langle u_0 \rangle$ and $H_1 = \left\langle u_1 + \frac{1}{2}u_0^2 \right\rangle$.

Assume now that the dynamical fields depend on an additional variable $y \in \mathbb{S}^1$. Then after the central extension procedure with the Maurer–Cartan 2-cocycle (56) the (2+1)-dimensional Lax hierarchy takes the form (58) [16]

$$L_{t_n} = [(\mathrm{d}C_n)_{\geqslant k-1}, L - \alpha\partial_y] = \pi_0\,\mathrm{d}C_n \qquad k = 1, 2,$$

for the Lax operator (96) with $N > 1$, where $\mathrm{d}C_n = \sum_{i=0}^{\infty} a_{n-i}\mathcal{E}^{n-i}$ are solutions to (57), $[\mathrm{d}C_n, L - \alpha\partial_y] = 0$. The Poisson tensor (59) is given by

$$\pi_0\,\mathrm{d}H = [L - \alpha\partial_y, (\mathrm{d}H)_{<k-1}] + ([\mathrm{d}H, L - \alpha\partial_y])_{<2-k},$$

and the Dirac reduction is not required.

Example 4.4. The case of $k = 1$. An example of finite field reduction. The Lax operator is given by $L = \mathcal{E} + u + v\mathcal{E}^{-1}$. Then for $(\mathrm{d}C_1)_{\geqslant 0} = \mathcal{E} + u$ we have

$$L_{t_1} = [(\mathrm{d}C_1)_{\geqslant 0}, L - \alpha\partial_y] = \pi_0\,\mathrm{d}H_1 = \pi_1\,\mathrm{d}H_0 \iff \begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \begin{pmatrix} (E-1)v + \alpha u_y \\ v(1 - E^{-1})u \end{pmatrix}.$$

The respective Poisson tensors are

$$\pi_0 = \begin{pmatrix} \alpha\partial_y & (E-1)v \\ v(1-E^{-1}) & 0 \end{pmatrix} \quad \text{and, if } \alpha = 0, \quad \pi_1 = \begin{pmatrix} Ev - vE^{-1} & u(E-1)v \\ v(1-E^{-1})u & v(E - E^{-1})v \end{pmatrix}.$$

The Hamiltonians are $H_1 = \left\langle v + \frac{1}{2}u^2 \right\rangle$ and $H_0 = \langle u \rangle$.

4.2. Algebra of δ-pseudo-differential operators

Consider a generalized derivative in the algebra $\mathcal{F}$ of dynamical fields with values in $\mathbb{K}$ given by a linear map $\Delta : \mathcal{F} \to \mathcal{F}$ that satisfies the generalized Leibniz rule

$$\Delta(uv) = \Delta(u)v + E(u)\Delta \tag{99}$$

for an algebra automorphism $E : \mathcal{F} \to \mathcal{F}$. If $E = 1$, then Δ is an ordinary derivative. According to (99) we define a generalized differential operator

$$\delta u = \Delta(u) + E(u)\delta. \tag{100}$$

In (100) and in all subsequent expressions, Δ and E act only on the nearest function on their right-hand side. As above, expression (100) is a counterpart of an ordinary differential operator ∂ such that $\partial u = u_x + u\partial$. Using (100) we have

$$\delta^{-1}u = E^{-1}u\delta^{-1} + \delta^{-1}\Delta^{\dagger}u\delta^{-1} = E^{-1}u\delta^{-1} + E^{-1}\Delta^{\dagger}u\delta^{-2} + E^{-1}\Delta^{\dagger 2}u\delta^{-3} + \cdots,$$

where $\Delta^{\dagger} := -\Delta E^{-1}$.

Hence, we can further define an algebra of δ-pseudo-differential operators

$$\mathfrak{g} = \mathfrak{g}_{\geqslant k} \oplus \mathfrak{g}_{<k} = \left\{\sum_{i\geqslant k} u_i\delta^i\right\} \oplus \left\{\sum_{i<k} u_i\delta^i\right\}, \tag{101}$$

where $u_i \in \mathcal{F}$. The above algebra is noncommutative and associative. The Lie structure on $\mathfrak{g}$ is given by the commutator $[A, B] = AB - BA$. In fact, we have the following result.

Proposition 4.5. *The algebra (101) generated by the rule of the form (100) is associative if and only if $\Delta : \mathcal{F} \to \mathcal{F}$ satisfies (99) and $E : \mathcal{F} \to \mathcal{F}$ is an algebra automorphism.*

We omit the proof, which can be done by induction, and it suffices to consider the associativity condition, $(AB)C = A(BC)$, only for $A = \delta^i$, $B = b\delta^j$ and $C = c$, where $a, b, c \in \mathcal{F}$.

In fact, we will consider only two most important special cases of generalized derivatives and δ-pseudo-differential operators:

- The first case is the ordinary derivative, i.e., (99) with $E = 1$. Thus let $\mathcal{F}$ consist of smooth functions $u_i : \Omega \to \mathbb{K}$ of x, where $\Omega = \mathbb{S}^1$ or $\Omega = \mathbb{R}$ (if u_i belong to the Schwartz space). Thus $\Delta = \partial_x$, and we let $\delta := \partial$. In this case, $\mathfrak{g}$ is the algebra of standard pseudo-differential operators with the trace form given by

$$\operatorname{tr} A = \langle \operatorname{res} A\rangle = \int_{\Omega} \operatorname{res} A \,\mathrm{d}x \qquad A = \sum_i a_i\partial^i \in \mathfrak{g}, \qquad \operatorname{res} A := a_{-1}. \tag{102}$$

- The second case is the generalized derivative given by a difference operator. Thus, let

$$\Delta = \frac{1}{\mu}(E - 1) \quad\Longrightarrow\quad \delta = \frac{1}{\mu}(\mathcal{E} - 1),$$

where μ is constant, E is an algebra automorphism and $\mathcal{E}$ is a shift operator such that (91) holds, see the realizations from the previous section. In particular, in the continuous lattice case:

$$\Delta f(x) = \frac{f(x+\hbar) - f(x)}{\hbar}, \qquad Ef(x) = f(x+\hbar), \qquad \mu = \hbar,$$

and in the q-deformed case:

$$\Delta f(x) = \frac{f(qx) - f(x)}{q-1}, \qquad Ef(x) = f(qx), \qquad \mu = q - 1.$$

Here the trace form is exactly the same as in the case of shift operators algebra from previous section (92). Thus, we have to consider the restriction (93) again, which is now equivalent to $\langle \Delta f\rangle = 0$. Note that δ-pseudo-differential operators can be represented uniquely by shift operators, with the convention that the δ-operators of negative orders are expanded into shift operators of negative orders as well. Note that in this case $\Delta^\dagger = -\Delta E^{-1}$.

In the quasi-classical limit, the algebra of standard pseudo-differential operators gives the Poisson algebra with canonical Poisson bracket (78) for $r = 0$. Thus, in dispersionless limit continuous soliton systems yield respective dispersionless systems. In the case when difference operator Δ is given on lattices, the continuous limit of the algebra of pseudo-δ-differential operators is actually given by the algebra of standard pseudo-differential operators. Thus, in contrast to the previous case, the quasi-classical limit of discrete soliton systems yields continuous soliton systems. This situation is similar in the q-deformed cases [12].

The reason why we consider both cases, i.e., continuous and discrete, is the fact that they can be unified into a single consistent scheme using the so-called timescales [12, 43, 97]. This scheme also includes soliton systems with spatial variable belonging to the space being a composition of continuous and discrete intervals. In [97], the trace formula on the algebra of pseudo-δ-differential operators is unified to the form that covers all the above realizations as well the cases with nonconstant μ.

Proposition 4.6. *The inner product (27) on $\mathfrak{g}$ defined by means of traces (102) and (92) in both cases is nondegenerate, symmetric and* ad*-invariant.*

Proof. In the first case of ordinary derivative nondegeneracy is obvious, the symmetricity follows from integration parts and the fact that integrals of total derivatives vanish. The ad-invariance is a consequence of lemma 2.8. In the second case of difference operators the proof follows from proposition 4.2 and the fact that δ-pseudo-differential operators can be expanded by means of shift operators. □

In general, the subspaces $\mathfrak{g}_{\geqslant k}$ and $\mathfrak{g}_{<k}$ are Lie subalgebras of $\mathfrak{g}$ (101) only for $k = 0$ and $k = 1$. However, if $E = 1$ they are Lie subalgebras also for $k = 2$. The classical R-matrices following from the decomposition of $\mathfrak{g}$ are $R = P_{\geqslant k} - \frac{1}{2}$. Hence, we have the following Lax hierarchies of commuting evolution equations:

$$L_{t_n} = \left[\left(L^{\frac{n}{N}}\right)_{\geqslant k}, L\right] = \pi_0\, \mathrm{d}H_n = \pi_1\, \mathrm{d}H_{n-1} \qquad k = 0, 1 \tag{103}$$

and for $k = 2$ if $E = 1$. The infinite-field Lax operators $L \in \mathfrak{g}$ generating (103) are given in the form

$$L = u_N\delta^N + u_{N-1}\delta^{N-1} + u_{N-2}\delta^{N-2} + u_{N-3}\delta^{N-3} + \cdots \qquad N \geqslant 1, \tag{104}$$

where in general for $k = 0$ the field u_N is time-independent; if $E = 1$ then for $k = 0$ the field u_{N-1} is also time independent and for $k = 1$ the same for the field u_N.

The explicit form of the differentials $\mathrm{d}H = \sum_i \delta^{-i-1}\gamma_i$ with respect to general Lax operator (104) has to be such that (16) is valid. See example 4.8. In the case of $E = 1$ we have $\gamma_i = \frac{\delta H}{\delta u_i}$. The Hamiltonians (29) are given by

$$H_n(L) = \frac{N}{N+n}\operatorname{tr}\left(L^{\frac{n}{N}+1}\right) \qquad N \neq -n.$$

We will consider the Hamiltonian structures of (103) only for $k = 0$. Thus, for $k = 0$ we have $R^* = -R$. Hence the linear Poisson tensor is given by (34),

$$\pi_0\, \mathrm{d}H = [(\mathrm{d}H)_{\geqslant 0}, L] - ([\mathrm{d}H, L])_{\geqslant 0}. \tag{105}$$

The case of quadratic Poisson tensor is more complex. For the pseudo-differential operators, when $E = 1$, the quadratic Poisson tensor is given by

$$\pi_1^{\rm red}\, \mathrm{d}H = (L\, \mathrm{d}H)_{\geqslant 0} L - L(\mathrm{d}H L)_{\geqslant 0} + \frac{1}{N}\left[\partial_x^{-1}(\operatorname{res}[\mathrm{d}H, L]), L\right], \tag{106}$$

after Dirac reduction applied to (38). In the case of difference operators ($E \neq 1$), as the decomposition for $k = 0$ of the algebra of δ-pseudo-differential operators (101) coincides with the decomposition of the algebra of purely shift operators (90) for $k = 1$, the quadratic Poisson tensor is given by (97), see [64, 97] for further details.

The pseudo-differential case ($E = 1$) of (103) with $k = 0$ and finite-field reductions, given by constraint $u_i = 0$ for $i < 0$, is the well-known Gelfand–Dickey hierarchy [40]. More details of the R-matrix formalism applied to the algebra of pseudo-differential operators for the remaining values of k can be found in [49] or [8].

Example 4.7. Consider the infinite-field case $k = 0$ of (104) in the ordinary derivative case $E = 1$, i.e., $L = \partial + \sum_{i=0}^{\infty} u_i \partial^{-i-1}$. This is the case of the well-known KP hierarchy [23]. Then we find the first nontrivial dispersive chain from (80),

$$L_{t_2} = [(L^2)_{\geqslant 0}, L] = \pi_0\, \mathrm{d}H_2 = \pi_1\, \mathrm{d}H_1$$

$$\iff (u_i)_{t_2} = (u_i)_{2x} + 2(u_{i+1})_x - 2\sum_{k=1}^{i}(-1)^k \binom{i}{k} u_{i-k}(u_0)_{kx},$$

where $(L^2)_{\geqslant 0} = \partial^2 + 2u_0$. The bi-Hamiltonian structure is given by linear

$$\pi_0^{ij} = \sum_{k=1}^{j}\binom{j}{k}\partial_x^k u_{i+j-k} - \sum_{k=1}^{i}(-1)^k\binom{i}{k} u_{i+j-k}\partial_x^k$$

and quadratic Poisson tensor

$$\begin{aligned}
\pi_1^{ij} = {}& \sum_{k=1}^{j+1}\binom{j+1}{k}\partial_x^k u_{i+j-k+1} - \sum_{k=1}^{i+1}(-1)^k\binom{i+1}{k} u_{i+j-k+1}\partial_x^k \\
&+ \sum_{l=0}^{j-1}\left[\sum_{k=1}^{j-l-1}\binom{j-l-1}{k} u_l \partial_x^k u_{i+j-k-l-1} - \sum_{k=1}^{i}(-1)^k\binom{i}{k} u_{i-k+l}\partial_x^k u_{j-l-1}\right] \\
&- \sum_{l=0}^{j-1}\sum_{k=0}^{i}\sum_{s=1}^{j-l-1}(-1)^k\binom{i}{k}\binom{j-l-1}{s} u_{i-k+l}\partial_x^{k+s} u_{j-l-s-1},
\end{aligned}$$

that are obtained from (105) and (106), respectively. The respective Hamiltonians are

$$H_1 = \int_{-\infty}^{\infty} u_1\, \mathrm{d}x \qquad H_2 = \int_{-\infty}^{\infty}\left(u_0^2 + u_2\right)\mathrm{d}x.$$

Example 4.8. The case of $k = 0$. In this example, we present the associated integrable systems and their Hamiltonian structures in the form that is valid in both continuous and discrete cases. In the first case, $E = 1$, $\mu = 0$ and $\Delta = -\Delta^{\dagger} = \partial_x$.

Let the Lax operator be given in the form

$$L = \delta + \mu\psi\varphi + \psi\delta^{-1}\varphi. \tag{107}$$

Then the first and the second flows from the Lax hierarchy (95) are

$$\begin{aligned}
\psi_{t_1} &= \mu\psi^2\varphi + \Delta\psi, \\
\varphi_{t_1} &= -\mu\varphi^2\psi - \Delta^{\dagger}\varphi
\end{aligned} \tag{108}$$

and

$$\begin{aligned}\psi_{t_2} &= \mu^2\psi^3\varphi^2 + 2\psi^2\varphi + \Delta^2\psi + \Delta(\mu\psi^2\varphi) + 2\mu\psi\varphi\Delta\psi + \mu\psi^2\Delta^\dagger\varphi\\ \varphi_{t_2} &= -\mu^2\psi^2\varphi^3 - 2\psi\varphi^2 - \Delta^{\dagger 2}\varphi - \Delta^\dagger(\mu\psi\varphi^2) - \mu\varphi^2\Delta\psi - 2\mu\psi\varphi\Delta^\dagger\varphi.\end{aligned} \tag{109}$$

For the Lax operator (107) the differential of a functional H such that (16) is valid is given by

$$\mathrm{d}H = \frac{1}{\varphi}\frac{\delta H}{\delta\psi} - \frac{1}{\psi}\Delta^\dagger\left(\frac{1}{\varphi}\right)\Delta^{-1}A - \delta\frac{1}{\psi\varphi + \mu\psi\Delta^\dagger\varphi}\Delta^{-1}A,$$

where $A = \psi\frac{\delta H}{\delta\psi} - \varphi\frac{\delta H}{\delta\varphi}$, and Δ^{-1} is a formal inverse of Δ. The linear and quadratic Poisson tensors take the form [97],

$$\pi_0 = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \qquad \pi_1 = \begin{pmatrix} -\mu\psi^2 - 2\psi\Delta^{-1}\psi & \Delta + 2\mu\psi\varphi + 2\psi\Delta^{-1}\varphi\\ -\Delta^\dagger + 2\varphi\Delta^{-1}\psi & -\mu\varphi^2 - 2\varphi\Delta^{-1}\varphi\end{pmatrix},$$

while the Hamiltonians are

$$\begin{aligned} &H_0 = \langle\psi\varphi\rangle, \qquad H_1 = \left\langle \tfrac{1}{2}\mu\psi^2\varphi^2 + \varphi\Delta\psi\right\rangle,\\ &H_2 = \left\langle \tfrac{1}{3}\mu^2\psi^3\varphi^3 + \psi^2\varphi^2 + \varphi\Delta^2\psi + \mu\psi\varphi^2\Delta\psi + \mu\psi^2\varphi\Delta^\dagger\varphi\right\rangle.\end{aligned}$$

In particular, when $E = 1, \mu = 0$ and $\Delta = \partial_x$ the above bi-Hamiltonian hierarchy is precisely the bi-Hamiltonian field soliton AKNS hierarchy [67]. In this case the first nontrivial flow is the second one (109), i.e., the AKNS system. When Δ is the difference operator, we obtain in particular the lattice [64] and the q-discrete counterparts of the AKNS hierarchy, where the first nontrivial flow is (108).

In fact, this example is more general and also includes more complex situations when μ is non-constant (time-independent) function on $\mathbb{R}$, for the details we send the reader to [97].

Let $\mathfrak{g}$ be the algebra of pseudo-differential operators (with respect to x), for which the coefficients depend on two independent spatial variables x and y. The central extension procedure with (56) yields the following Lax hierarchy (58) [16]:

$$L_{t_n} = [(\mathrm{d}C_n)_{\geqslant k}, L - \alpha\partial_y] = \pi_0\,\mathrm{d}C_n,$$

where $k = 1, 2$ or $k = 3$ and $L = u_N\partial^N + u_{N-1}\partial^{N-1} + u_{N-2}\partial^{N-2} + \cdots$, with $u_N = 1, u_{N-1} = 0$ for $k = 0$ and only $u_N = 1$ if $k = 1$. The Lax hierarchies are generated by $\mathrm{d}C_n = \sum_{i=0}^{\infty} a_{n-i}\partial^{n-i}$ solving (57), i.e. $[\mathrm{d}C_n, L - \alpha\partial_y] = 0$. The associated Poisson tensor (59) is given by

$$\pi_0\,\mathrm{d}H = [(\mathrm{d}H)_{\geqslant 0}, L - \alpha\partial_y] - ([\mathrm{d}H, L - \alpha\partial_y])_{\geqslant 0},$$

and requires no Dirac reduction.

Example 4.9. The case of $k = 0$ with the Lax operator of the form $L = \partial^2 + u$. Then, for $(\mathrm{d}C_3)_{\geqslant 0} = \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}\left(u_x + \alpha\partial_x^{-1}u_y\right)$ we obtain the (2+1)-dimensional KP equation

$$L_{t_3} = [(\mathrm{d}C_3)_{\geqslant 0}, L - \alpha\partial_y] = \pi_0\,\mathrm{d}H_3 = \pi_1\,\mathrm{d}H_1 \quad\Longleftrightarrow\quad u_{t_3} = \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}uu_x + \tfrac{3}{4}\alpha^2\partial_x^{-1}u_y,$$

where the Poisson tensors are $\pi_0 = 2\partial_x$ and, if $\alpha = 0$, $\pi_1 = \frac{1}{2}\partial_x^3 + \partial_x u + u\partial_x$. The Hamiltonians are

$$H_1 = \iint_{\Omega\times\mathbb{S}^1} \frac{1}{4}u^2\,\mathrm{d}x\,\mathrm{d}y \qquad H_3 = \iint_{\Omega\times\mathbb{S}^1} \frac{1}{16}\left(2u^3 + uu_{xx} + 3\alpha^2 u\partial_x^{-2}u_{yy}\right)\mathrm{d}x\,\mathrm{d}y.$$

More examples of (2+1)-dimensional field systems can be found in [16, 93], where one can find also systems that are purely (2+1)-dimensional phenomena.

4.3. $sl(2,\mathbb{C})$ loop algebra

We will follow the scheme from section 2.10. Consider a loop algebra over the classical Lie algebra $\mathfrak{g} = sl(2,\mathbb{C})$ of traceless 2×2 nonsingular matrices, i.e., $\mathfrak{g}^\lambda = sl(2,\mathbb{C})[\![\lambda,\lambda^{-1}]\!]$, with coefficients being smooth dynamical functions of variable $x \in \Omega$. The space $u_i : \Omega \to \mathbb{K}$ are smooth functions of x, where $\Omega = \mathbb{S}^1$ if we assume these functions to be periodic in x or $\Omega = \mathbb{R}$ if these functions belong to the Schwartz space. The commutator defines the Lie bracket in $sl(2,\mathbb{C})$ and readily extends to $\mathfrak{g}^\lambda$ since (62).

We already know that there are two natural decompositions of $\mathfrak{g}^\lambda$ into Lie subalgebras (63), and we consider only the one for $k = 0$ yielding the classical R-matrix $R = P_+ - \frac{1}{2}$, with P_+ being the projection onto the nonnegative powers of λ.

The trace form on $\mathfrak{g}^\lambda$ is (69), $\operatorname{Tr}(a) = \int_\Omega \operatorname{res}\operatorname{tr}(a)\,\mathrm{d}x$, where $\operatorname{res}\sum_i a_i\lambda^i = a_{-1}$ and tr is the standard trace of matrices. As the matrices in $sl(2,\mathbb{C})$ are nonsingular, the trace tr defines nondegenerate inner product which is also symmetric and ad-invariant. These properties extend to $\mathfrak{g}^\lambda$, and hence $\mathfrak{g}^{*\lambda} \cong \mathfrak{g}^\lambda$ and $\mathrm{ad}^* \equiv \mathrm{ad}$.

Applying the central extension procedure with the Maurer–Cartan 2-cocycle (56) ($x \equiv y$) yields the Lax hierarchy given by (74) (we take $\alpha = 1$), i.e.,

$$L_{t_n} = [(\mathrm{d}C_n)_+, L - \partial_x] = \cdots = \pi_l\,\mathrm{d}C_{n-l} = \cdots \tag{110}$$

for the Lax operators

$$L = \boldsymbol{u}_N\lambda^N + \boldsymbol{u}_{N-1}\lambda^{N-1} + \cdots + \boldsymbol{u}_{1-m}\lambda^{1-m} + \boldsymbol{u}_{-m}\lambda^{-m} \qquad N \geqslant -1, \tag{111}$$

where $\boldsymbol{u}_i \in sl(2,\mathbb{C})$ and $\boldsymbol{u}_N$ is a constant matrix. The Lax hierarchy (110) is generated by $\mathrm{d}C_n = \lambda^n\,\mathrm{d}C_0$ such that $\mathrm{d}C_0 = \sum_{i=0}^\infty a_i\lambda^i$, where $a_i \in sl(2,\mathbb{C})$, satisfies $[\mathrm{d}C_0, L - \partial_x] = 0$.

For a given functional $H \in \mathcal{C}^\infty(\mathfrak{g}^\lambda)$ of (111) its differential has the form

$$L = \frac{\delta H}{\delta \boldsymbol{u}_{-m}}\lambda^{m-1} + \frac{\delta H}{\delta \boldsymbol{u}_{1-m}}\lambda^{m-2} + \cdots + \frac{\delta H}{\delta \boldsymbol{u}_{N-1}}\lambda^{-N},$$

where

$$\frac{\delta H}{\delta \boldsymbol{u}} = \begin{pmatrix} \frac{1}{2}\frac{\delta H}{\delta u_{11}} & \frac{\delta H}{\delta u_{21}} \\ \frac{\delta H}{\delta u_{12}} & -\frac{1}{2}\frac{\delta H}{\delta u_{11}} \end{pmatrix} \quad \text{for} \quad \boldsymbol{u} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix}.$$

The multi-Hamiltonian structure for (110) is given by (77)

$$\pi_l\,\mathrm{d}H = [(\lambda^l\,\mathrm{d}H)_+, L - \partial_x] - \lambda^l[\mathrm{d}H, L - \partial_x]_+. \tag{112}$$

For the general Lax operators (111), if $N \geqslant l \geqslant -m$ for $N \geqslant 0$ and $0 \geqslant l \geqslant -m$ for $N = -1$ then the Dirac reduction of (112) is not required.

Example 4.10. Consider the Lax operator of the form

$$L = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}\lambda + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \qquad \mathrm{i} = \sqrt{-1}. \tag{113}$$

Then we find that

$$\mathrm{d}C_0 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\lambda^{-1} + \begin{pmatrix} -\frac{\mathrm{i}}{2}rq & \frac{\mathrm{i}}{2}q_x \\ -\frac{\mathrm{i}}{2}r_x & \frac{\mathrm{i}}{2}rq \end{pmatrix}\lambda^{-2} + \cdots.$$

Hence, from (110) we obtain the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy [1]

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_1} = \begin{pmatrix} q_x \\ r_x \end{pmatrix}, \quad \begin{pmatrix} q \\ r \end{pmatrix}_{t_2} = \begin{pmatrix} \frac{\mathrm{i}}{2}q_{xx} - \mathrm{i}rq^2 \\ -\frac{\mathrm{i}}{2}r_{xx} + \mathrm{i}r^2q \end{pmatrix}, \quad \begin{pmatrix} q \\ r \end{pmatrix}_{t_3} = \begin{pmatrix} -\frac{1}{4}q_{xxx} + \frac{3}{2}rqq_x \\ -\frac{1}{4}r_{xxx} + \frac{3}{2}rqr_x \end{pmatrix}, \ldots.$$

The bi-Hamiltonian structure is given by the Poisson tensors (112)

$$\pi_0 = \begin{pmatrix} 0 & -2\mathrm{i} \\ 2\mathrm{i} & 0 \end{pmatrix} \qquad \pi_1^{\mathrm{red}} = \begin{pmatrix} 2q\partial_x^{-1}q & \partial_x - 2q\partial_x^{-1}r \\ \partial_x - 2r\partial_x^{-1}q & 2r\partial_x^{-1}r \end{pmatrix}$$

with the hierarchy of Hamiltonians

$$H_1 = \int_\Omega \frac{\mathrm{i}}{2} q_x r \, \mathrm{d}x, \quad H_2 = \int_\Omega \frac{1}{4}(q^2r^2 - q_{xx}r) \, \mathrm{d}x, \quad H_3 = \int_\Omega \frac{\mathrm{i}}{8}(3qq_xr^2 - q_{xxx}r) \, \mathrm{d}x, \ldots .$$

Note that the need for the Dirac reduction for the Poisson tensor π_1^{red} follows from the fact that (113) is not the most general Lax operator (111) from $\mathfrak{g}^\lambda$ with $N = 1$ and $m = 0$.

The reduction $q = \psi, r = \psi^*$ of the above AKNS hierarchy yields the nonlinear Schrödinger hierarchy, while the reductions $q = \mathrm{i}u, r = \mathrm{i}$ and $q = r = \mathrm{i}v$ give rise to the KdV and the mKdV hierarchies, respectively.

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